RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

CONVOLUTION OF SEQUENCES[1]

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A summability method is a linear functional on a space of sequences $\mathcal{S}$. The method ϕ is said to be regular if, for each convergent sequence $s=\{s_n\}$ we have $\phi(s)=\lim_{n\to\infty} s_n$. In this paper we define various types of convolution (multiplication) of sequences; we use the symbol $*$ to denote convolution. Our convolution is always distributive, but not necessarily associative or commutative. We consider the regular methods ϕ such that $\phi(s*t)=\phi(s)\phi(t)$ for all sequences s and t in the domain of ϕ, $\mathcal{S}(\phi)$, that is, the regular homomorphisms from $\mathcal{S}(\phi)$ to the real numbers. We write $\hat{s}(\phi)=\phi(s)$ for each sequence s in $\mathcal{S}(\phi)$ and we impose the weak topology on the set of homomorphic methods. In case the multiplication is commutative and associative and we were dealing with complex sequences, then $\mathcal{S}(\phi)$ would be a complex Banach algebra, $\hat{s}(\phi)$ would be the Fourier transform of the sequence s, and the weak topology on the set of homomorphic methods would yield the maximal ideal space of $\mathcal{S}(\phi)$. Although we shall deal with real sequences, we shall use a certain amount of Gel'fand theory.

The types of convolution to be considered are:

(a) Pointwise multiplication—if s and t are two bounded sequences then $s*t=\{s_nt_n\}$.

(b) Cauchy multiplication—if s and t are two sequences such that

$$S(z) = \sum_{n=0}^{\infty} a_n z^n, \quad T(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (a_n = s_{n+1} - s_n,\ b_n = t_{n+1} - t_n)$$

are analytic and bounded in the unit circle D in the complex z-plane, then $s*t=\{\sum_{k=0}^{n}\sum_{j=0}^{k} a_j b_{k-j}\}$. We note that the power corresponding to $s*t$ is $S(z)T(z)$.

(c) If s and t are bounded sequences, and $B=(b_{nk})$ is a positive

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regular triangular summation matrix, we define $s * t = \{ \sum_{k=0}^{n} b_{nk}s_k t_k \}$.

(d) If s and t are two bounded sequences, B has all properties stated in (c), and, in addition,

$$\lim b_{n,n-r} = 0, \qquad r = 0, 1, \cdots,$$

then

$$s * t = \left\{ \sum_{k=0}^{n} b_{nk}s_k t_{n-k} \right\}.$$

Convolutions (a) and (b) are commutative and associative, convolution (c) is commutative but not associative while convolution (d) is neither commutative nor associative. If ϕ is a regular homomorphism relative to (a), (c), or (d) we turn $\mathcal{S}(\phi)$ into a Banach space by imposing the norm $\|s\| = \sup |s_n|$; if ϕ is a homomorphism relative to (b) we use the norm $\|s\| = \sup S(z)$, the supremum being taken over all points z in D.

THEOREM 1A. *If ϕ is a regular homomorphism relative to* (a) *or* (c), *and s is in $\mathcal{S}(\phi)$, then $\phi(s)$ is a cluster value of s.*

We first show that lim inf $s \leqq \phi(s) \leqq$ lim sup s. If $\phi(s) = \sigma >$ lim sup s then lim sup $[s/\sigma]^{(m)} \to 0$ as $m \to \infty$ (here $s^{(m)}$ denotes the sequence s convolved with itself m times). We use the fact that ϕ is a linear continuous functional to conclude that $\phi(s/\sigma)^{(m)} \to 0$. Since ϕ is a homomorphism, we must have $\phi(s/\sigma^{(m)}) = 1$ for all m. We have a contradiction; thus $\phi(s) \leqq$ lim sup s. Similarly we see that $\phi(s) \geqq$ lim inf s. In particular ϕ must evaluate the sequence $(s-\sigma)^{(2)}$ to 0, since $(s-\sigma)^{(2)}$ is a non-negative sequence when the convolution considered is (a) or (c), 0 must be a cluster value of $s-\sigma$. In other words, σ must be a cluster value of $\phi(s)$.

THEOREM 1B. *If ϕ is a regular homomorphism relative to* (b), *and $s \in \mathcal{S}(\phi)$ satisfies*

$$\sup |s_n| \leqq M \sup_{z \in D} |S(z)| \tag{1}$$

for some constant M, then $\phi(s)$ is a cluster value of $S(z)$ as $z \to 1-$.

THEOREM 1C. *If ϕ is a regular homomorphism relative to* (d), *then* lim inf $s_n \leqq \phi(s) \leqq$ lim sup s_n, *for each sequence s in $\mathcal{S}(\phi)$.*

When ϕ is a homomorphism relative to (d), we cannot imitate the proof of Theorem 1A to conclude that $\phi(s)$ must be a cluster value of s; with this convolution $s * s$ may be negative.

THEOREM 2A. *Suppose that the sequence s satisfies* (1). *If s is Abel* *summable and evaluated by some method* ϕ *which is a regular homomorphism relative to* (b), *then* $\phi(s)$ *must equal the Abel sum of* s.

THEOREM 2B. *If the bounded sequence* s *is evaluated to* σ *by the matrix* B *and it is in* $\mathcal{S}(\phi)$, *where* ϕ *is a regular homomorphism relative to* (c) *or* (d), *then* $\phi(s)=\sigma$.

Theorem 2A follows from Theorem 1B; to prove Theorem 2B we note that if the matrix B evaluates s to σ, then $s * 1 \rightarrow \sigma$.

THEOREM 3A. *If* ϕ *is a regular homomorphism relative to convolution* (a) *or* (c), *and* s *is a sequence in* $\mathcal{S}(\phi)$ *which is bounded away from* 0, *then* $\{1/s_n\}$ *is in* $\mathcal{S}(\phi)$; *if* s *and* t *are in* $\mathcal{S}(\phi)$, *then the sequences* $s \vee t = \max(s_n, t_n)$ *and* $s \wedge t = \min(s_n, t_n)$ *are in* $\mathcal{S}(\phi)$.

If ϕ is a regular homomorphism relative to (a) or (c), $s \in \mathcal{S}(\phi)$, and $\phi(s)=\sigma$, then $(s-\sigma)^{(2)}$ is a non-negative sequence which ϕ evaluates to 0. Consequently, if ϵ is a positive number, the set of integers n, on which $|s_n-\sigma| > \epsilon$ is sparse. The same must be true for the set of integers on which $|1/s_n - 1/\sigma| > \epsilon$ and $\phi(\{1/s_n\}) = 1/\sigma$.

To show that $s \vee t$ is in $\mathcal{S}(\phi)$, we note that if $\phi(s)=\sigma$, $\phi(t)=\tau$, then there exist subsequences $\{s_{n_j}\}$, $\{t_{m_j}\}$ which converge to σ and τ; moreover the sequences of integers $\{n_j\}$ and $\{m_j\}$ are fairly dense. Consequently, the sequence $\{n_j\} \cap \{m_j\}$ is also fairly dense and $s \vee t$ has a subsequence converging to $\max(\sigma, \tau)$ along this intersection. Hence $\phi(s \vee t) = \max[\phi(s), \phi(t)]$, and similarly $\phi(s \wedge t) = \min[\phi(s), \phi(t)]$.

This theorem could have been proved by Banach algebra theory in the case where ϕ is a regular homomorphism relative to (a). By such a method we can prove:

THEOREM 3B. *If* ϕ *is a homomorphism relative to* (b), *and* s *is a sequence in* $\mathcal{S}(\phi)$ *such that the corresponding power series* $S(z)$ *is bounded away from* 0 *and* (1) *is satisfied, then the sequence corresponding to* $1/S(z)$ *is in* $\mathcal{S}(\phi)$.

THEOREM 4. *If* ϕ *is a regular homomorphism relative to* (b), *and* $\{s_n\}$ *is a sequence in* $\mathcal{S}(\phi)$, *then* $\{s_{n+1}\}$ *is in* $\mathcal{S}(\phi)$ *and* $\phi(\{s_{n+1}\}) = \phi(s_n)$.

Let ϕ_0 be a regular homomorphism and let Φ denote the set of all homomorphisms ϕ such that $\mathcal{S}(\phi) \supseteq \mathcal{S}(\phi_0)$. According to the weak topology a regular homomorphism is in the closure of a set $\{\phi_\alpha\}$ if and only if $s(\phi_0)$ is a cluster value of the set $\{\hat{s}(\phi_\alpha)\}$ for each s in the common convergence field. We denote the topological spaces formed by Φ_a, Φ_b, Φ_c, Φ_d, according as the convolution is (a), (b), (c) or (d).

THEOREM 5A. *The spaces Φ_a and Φ_c are totally disconnected.*

The proof depends on the fact that if ϕ is a regular homomorphism relative to (a) or (c), then each sequence in $\mathcal{S}(\phi)$ has a very dense subsequence which converges to $\phi(s)$.

Now suppose that s is a sequence such that the corresponding power series $S(z)$ is analytic in $|z|<1$. If σ is a number between $\limsup_{z\to 1-} S(z)$ and $\liminf_{z\to 1-} S(z)$, then there exists a sequence of points $\{z_n\}$ such that $z_n \to 1-$ and $S(z_n)\to\sigma$. The functional $\phi(s) = \lim_{z_n\to 1-} S(z_n)$ is a regular homomorphism relative to (b). In other words, for regular homomorphisms relative to (b), $\mathcal{S}(\phi)$ takes on each value between its upper and lower bound. Thus

THEOREM 5B. *The space Φ_b contains a continuum.*

The following is an example of a totally disconnected space Φ_d. Let the matrix $B=(b_{nk})$, defining the convolution, be given by

$$b_{n,n/2} = 1, \quad b_{nk} = 0, \quad k \neq n/2, \quad n \text{ even},$$
$$b_{n,k} = 1/(n+1), \quad k \leqq n, \quad b_{n,k} = 0, \quad k > n, \quad n \text{ odd}.$$

Let the method ϕ_0 be defined by the matrix $A=(a_{nk})$ where

$$a_{n,n} = 1, \quad a_{n,k} = 0, \quad k \neq n, \quad n \text{ even},$$
$$a_{n,n-1} = 1, \quad a_{n,k} = 0, \quad k \neq n-1, \quad n \text{ odd}.$$

The set of regular homomorphisms ϕ such that $\mathcal{S}(\phi)\supseteq\mathcal{S}(A)=\mathcal{S}(\phi_0)$ forms a totally disconnected space Φ_d under our weak topology.

I do not know whether spaces Φ_d containing a continuum exist.

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