ON THE TOPOLOGY OF RIEMANNIAN MANIFOLDS WHERE THE CONJUGATE POINTS HAVE A SIMILAR DISTRIBUTION AS IN SYMMETRIC SPACES OF RANK 1

BY WILHELM KLINGENBERG

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1. Manifolds similar to spheres.

1.1. Let $S=S^n$ be the n-dimensional sphere, endowed with the usual metric of constant Riemannian curvature 1. Let $G=(p(s))$, $0 \leqq s < \infty$, be a geodesic ray in S^n, s being the arc length. Then the conjugate points of $p(0)$ on G occur at $s=l\pi$, l a positive integer, with multiplicity $n-1$.

Let G be a geodesic ray in a Riemannian manifold $M=M^n$ of dimension n. The following condition may be interpreted, at least for $k=n-1$, as saying that the first k conjugate points on G are similarly distributed as on the sphere S^n:

(Σ, k) *There are no conjugate points in the interval* $[0, \pi[$ *and at least* k *conjugate points in* $[\pi, 2\pi[$, *each counted by its multiplicity.*

1.2. The following proposition gives a sufficient, but not necessary condition for the validity of (Σ, $n-1$). For the proof see Morse [5].

PROPOSITION 1. *Let* $G=(p(s))$ *be a geodesic ray in a Riemannian manifold* M^n. *Assume that the Riemannian curvature* $K(\sigma)$ *of a plane section* σ, *tangent to* G *at a point* $p(s)$ *with* $s \leqq 2\pi$ *satisfies*

$$1/4 < K(\sigma) \leqq 1. \tag{1}$$

Then (Σ, $n-1$) *holds for* G.

1.3. We now study the implications of (Σ, k):

LEMMA 1. *Let* $M=M^n$ *be a simply connected, complete Riemannian manifold and assume that there is a point* $o \in M$ *such that for each geodesic ray, starting at* o, (Σ, k) *holds with* $k \geqq 2$. *Then* M *is compact. There is a point* $o' \in M$ *with* $d(o, o') \sim 0$ *and not conjugate to* o *such that each geodesic from* o *to* o' *which is not the geodesic of minimal length* $d(o, o')$ *has length* $\geqq 2\pi - d(o, o') \sim 2\pi$ *and, therefore, has an index* $\geqq k$.

Hence, the loop space $\Omega(o, o')$ *has the homotopy type of a* 0*-cell to which there are attached cells of dimension* $\geqq k$.

The proof of this lemma goes along the same lines as the proof of the lemma in [4] except that Rauch's comparison theorem is replaced by an application of the Gauss lemma.

1.4. Using standard facts in Morse theory we have

THEOREM 1. *If M is an n-dimensional, simply connected, complete Riemannian manifold with the property that there is a point $o \in M$ such that each geodesic ray starting from o satisfies (Σ, k), with $k \geqq 2$, then $\pi_i(M) = 0$ for $1 \leqq i \leqq k$.*

If here $(n-1)/2 \leqq k$, then Poincaré duality and standard facts in homotopy theory yield that M has the homotopy type of the sphere and hence, according to Smale [7], M is actually homeomorphic with the sphere, at least for $n \neq 3$ and $n \neq 4$.

Combined with Proposition 1 we get

THEOREM 2. *Let M^n be a complete, simply connected Riemannian manifold. If there is a point $o \in M$ such that for each plane section σ, tangent to one of the geodesic segments of length 2π emanating from* 0, *the Riemannian curvature $K(\sigma)$ satisfies* (1), *then M^n is compact and has the homotopy type of the sphere and even is, at least for $n \neq 3$, 4, homeomorphic with the sphere.*

1.5. Presumably, under the assumptions of Theorem 2, M^n is homeomorphic to the sphere for all n. At least, this is the case when the assumptions do hold for all $o \in M$ or, what is the same, if (1) holds for all plane sections σ of M. As this follows was shown in [4], from an argument provided by Berger [1]. A variation of this argument was given independently by Toponogov [9]. Both proofs are based on the information on the length of closed geodesics as provided by Lemma 1 and on Toponogov's theorem on geodesic triangles (cf. [8]). Tsukamoto [10] gave a third proof in which only an infinitesimal version of the triangle theorem is used which is due to Rauch [6].

2. **Manifolds similar to one of the other compact, irreducible symmetric spaces of rank 1.**

2.1. Recall that these spaces belong to one of the following classes (cf. Cartan [2]):

The complex projective space, $P(1)^n$, having a dimension $n = 2m \geqq 4$.

The quaternion projective space, $P(3)^n$, having a dimension $n = 4m \geqq 8$.

The projective Cayley plane, $P(7)^n$, having the dimension $n = 16$.

These spaces shall be endowed with their usual metric in which the values of the Riemannian curvature vary between 1 and 1/4. Let $G = (p(s))$ be a geodesic ray in the space $P(\alpha)^n$, $\alpha \in \{1, 3, 7\}$. Then the conjugate points of $p(0)$ on G occur at $s=(2l-1)\pi$, l a positive integer, with multiplicity α, and at $s=2l\pi$, l a positive integer, with multiplicity $n-1$.

2.2. The following condition may be paraphrased by saying that the distribution of the conjugate points is, to the given degree, similar to the distribution which occurs in the symmetric space $P(\alpha)^n$, $\alpha\in\{1, 3, 7\}$: $\Pi(\alpha, n)$. *There are no conjugate points in the intervals* $[0, \pi[$ *and* $[5\pi/4, 2\pi[$, *there are* α *conjugate points in* $[\pi, 5\pi/4[$ *and there are* $n-1$ *conjugate points in* $[2\pi, 5\pi/2[$, $n=(\alpha+1)m$, *each counted with its multiplicity.*

2.3. Let o be a point in a complete Riemannian manifold $M=M^n$. Consider the exponential map exp: $M_0\rightarrow M$ of the tangent space M_0 of M at o onto M. To each ray $\overline{G}=(\bar{p}(s))$, $0\leqq s<\infty$, in M_0, starting from the origin $o\in M_0$, there corresponds the geodesic ray $G=(p(s))$, $0\leqq s<\infty$, in M, starting from $o\in M$ in the same direction as G. Then $\bar{p}(s)$ is a critical point for the exponential map if and only if $p(s)$ is a conjugate point on G.

We use this well-known fact to explain what it means that $\Pi(\alpha, n)$ holds for all geodesic rays starting from o; later we will see that this property has far reaching consequences for the topology of M.

Denote by $B(s)$ the open ball in M_0 of radius s and center at the origin $o\in M_0$. Then our assumption implies, first of all, that there are no critical points for exp in $B(\pi)$. In contrast, a ray $\overline{G}$ passing through $D=B(5\pi/4)-B(\pi)$, will hit α critical points; we like to think, therefore, of D as of some sort of van Allen radiation belt. But once we are beyond this belt we reach again a safe region $E=B(2\pi)-B(5\pi/4)$ without critical points. The far side of E, however, is again surrounded by a dangerous belt, i.e., $B(5\pi/2)-B(2\pi)$, which is thickly populated ($n-1$ on each ray!) with critical points.

It is the safe belt E, beyond the first dangerous belt D, which constitutes the essential new feature compared with the situation considered in Chapter 1.

Note that for the symmetric space $P(\alpha)^n$ the two dangerous belts are squeezed together into spheres of radius π and 2π, respectively.

The condition $\Pi(\alpha, n)$ may be interpreted as a certain perturbation of this highly degenerate and unstable situation. Of course, one may consider even stronger perturbations than the one described by $\Pi(\alpha, n)$. However, the results we are able to draw in such a case are less conclusive than the one presented below.

2.4. The following proposition gives a sufficient but not necessary condition for the validity of $\Pi(1, n)$ in a Kähler manifold $M=M^n$. First we recall that a plane section σ in M determines an angle

$\omega = \omega(\sigma)$, $0 \leqq \omega \leqq \pi/2$, in the following way: If X is a vector $\neq 0$ in σ, let $\bar{\sigma}$ be the 2-plane spanned by X and JX, J being the imaginary operator; then $\omega(\sigma)$ is defined as the angle between σ and $\bar{\sigma}$.

For the complex projective space $P(1)$ the Riemannian curvature $K(\sigma)$ of a plane section σ depends only on the angle $\omega = \omega(\sigma)$ and is given by $K_1(\omega) = (1 + 3\cos^2 \omega)/4$.

PROPOSITION 2. *Let M^n be a Kähler manifold. Let $G = (p(s))$ be a geodesic ray in M^n. Assume that the Riemannian curvature $K(\sigma)$ of a plane section σ, tangent to G at a point $s \leqq 2\pi$ satisfies*

$$0.64\ K_1(\omega(\sigma)) < K(\sigma) \leqq K_1(\omega(\sigma)). \tag{2}$$

Then $\Pi(1, n)$ *holds for* G.

The proof follows from the index theorem of Morse [5].

2.5. We now give the implications of $\Pi(\alpha, n)$, $\alpha \in \{1, 3, 7\}$.

LEMMA 2. *Let M^n be a simply connected, complete Riemannian manifold and assume that there is a point $o \in M$ such that $\Pi(\alpha, n)$ holds for each geodesic ray starting from o. For $\alpha = 1$ assume, in addition, that there is a point in M which has distance π from o. We note that this is the case if M has positive Riemannian curvature for all plane sections* (cf. [3]). *Then the following does hold*:

(i) *M is compact.*

(ii) *There is a point $o' \in M$ with $d(o, o') \sim 0$ and not conjugate to o such that the loop space $\Omega(o, o')$ contains no geodesic of index i with $0 < i < \alpha$.*

(iii) *There is a point $o'' \in M$ with $d(o, o'') \sim \pi$ and not conjugate to o such that the loop space $\Omega(o, o'')$ contains only geodesics which either have length $< 5\pi/2 - d(o, o'') \sim 3\pi/2$ and hence have an index $\leqq \alpha$ or have length $> 3\pi/2 + d(o, o'') \sim 5\pi/2$ and hence have an index $\geqq n - 1 + \alpha$. Furthermore, the subspace $\Omega^{2\pi}$ of $\Omega(o, o')$, formed by the curves of length $\leqq 2\pi$ (which contains only geodesics of index $\leqq \alpha$) has the homotopy type of the α-sphere.*

(iv) *The loop space of M has the homotopy type of an α-sphere to which there are attached cells of dimension $> n - 2 + \alpha$.*

(i) is clear. (ii) follows from Lemma 1 which, for $\alpha = 3, 7$, also yields the existence of a point with distance π from o.

To prove (iii) we introduce the subspace $\tilde{\Omega} = \tilde{\Omega}(o, o'')$ of $\Omega = \Omega(o, o'')$ consisting of those curves which start out from o as a geodesic segment of length $5\pi/4$ and then continue to o''. On $\tilde{\Omega}$ we consider the length function. Then there are two types of critical points: First, the geodesic segments from o to o'' of length $\geqq 5\pi/4$; their index in

$\tilde{\Omega}$ is α units less than it is in Ω. Second, there are the once broken geodesic segments of the following form: They start as a geodesic from o to o'' of length $<5\pi/4$ and then go on beyond o'' until they reach the length $5\pi/4$ and then they return the same way back to o''. The index in $\tilde{\Omega}$ of such a critical point is given by the number of conjugate points on the initial segment of length $5\pi/4$ which occur after o''. That means: If the initial segment from o to o'' of length $<5\pi/4$ has index i in Ω, the broken segment has index $\alpha-i$ in $\tilde{\Omega}$.

The statements in (iii) on the length of geodesics in $\Omega(o, o'')$ are now proved with the help of a lifting argument for a homotopy between two critical points in $\tilde{\Omega}$, similar to the one used in the proof of Lemma 1. This time, however, the lifting of the curves of $\tilde{\Omega}$ into M_0 does not give curves in $B(\pi)$ but gives curves which start out from $o\in M_0$ as a straight segment of length $5\pi/4$ which brings them into the safe region $E=B(2\pi)-B(5\pi/4)$ (cf. 2.3) where they then continue to stay until either they fall back into the van Allen belt D (cf. 2.3) which is the uninteresting case or until they reach the outer border of E at a distance 2π.

The last statement in (iii) is proved by noting that $\Omega^{2\pi}$ and $\tilde{\Omega}^{2\pi}$ have the same homotopy type but yield CW-complexes which are dual to each other. Standard facts then give (iv).

2.6. Property (iv) in Lemma 2 implies that M^n and $P(\alpha)^n$ have the same homotopy groups up to dimension $n-1+\alpha$. A spectral sequence argument gives that M^n and $P(\alpha)^n$ have the same integer cohomology ring. Combining this with Proposition 2 we get

THEOREM 3. *Let M^n, $n\geqq 4$, be a complete Kähler manifold and assume that the Riemannian curvature $K(\sigma)$ of the plane sections σ of M^n satisfies* (2). *Then M^n has the homotopy type of the complex projective space $P(1)^n$.*

Indeed, since $\pi_i(P(1)^n)=0$ for $3\leqq i\leqq n$, it is possible to extend a map of the 2-skeleton of M into $P(1)$ to a map of M into $P(1)$. In particular, this can be done such as to imply an isomorphism of the cohomology ring. But then M and $P(1)$ have the same homotopy type (cf. Whitehead [11]).

On the other hand, for $\alpha=3, 7$, we have at least

THEOREM 4. *Let M be a simply connected, complete Riemannian manifold of the same dimension as the symmetric space $P(\alpha)^n$, $\alpha\in\{3, 7\}$. If there is a point $o\in M$ such that $\Pi(\alpha, n)$ holds for each geodesic ray starting at o then M is compact and has the same integer cohomology ring as $P(\alpha)^n$.*

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UNIVERSITY OF CALIFORNIA, BERKELEY