RESTRICTION OF ISOTOPIES

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Let M be a connected and simply connected topological manifold (with or without boundary) and m a fixed point in Int M, the interior of M. Let h_0 and h_1 be two isotopic homeomorphisms of M, each of which leaves m fixed.

It is the object of this note to show that, under these conditions, $h_0/M-m$ and $h_1/M-m$ are isotopic homeomorphisms of $M-m$.

With the standard definition of isotopy, the result follows immediately from the covering homotopy theorem, but with a somewhat more liberal (and frequently more natural) definition of isotopy, the argument is less direct. In fact in this case I can obtain the result only with the aid of the apparently irrelevant assumption that M can support a piecewise linear structure.

The converse question of extending isotopies on a space to isotopies on its one-point compactification has already been answered affirmatively by R. H. Crowell [1] in the much more general setting of locally compact Hausdorff spaces.

1. **Definitions.** If h_0 and h_1 are homeomorphisms of X onto Y, an *isotopy* between h_0 and h_1 is a continuous map

$$H: X \times [0, 1] \rightarrow Y \times [0, 1]$$

such that

(i) $H(x, 0) = (h_0(x), 0)$ for all $x \in X$,

(ii) $H(x, 1) = (h_1(x), 1)$ for all $x \in X$,

(iii) $H/X \times t$ is a homeomorphism of $X \times t$ onto $Y \times t$ for all $t \in [0, 1]$.

It is shown in [1] that if X is a locally compact Hausdorff space, then condition (iii) above implies

(iii′) H is a homeomorphism.

H is called a *weak isotopy* between h_0 and h_1 if H satisfies conditions (i), (ii) and (iii′). Thus if X is locally compact and Hausdorff (in particular, if X is a manifold), an isotopy is also a weak isotopy, so that isotopic homeomorphisms will also be weakly isotopic.

Weak isotopy is an important notion in the study of topological manifolds. For example, the extendability of a homeomorphism de-

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fined on the boundary of a manifold to a homeomorphism of the whole manifold depends only on the weak isotopy class of the homeomorphism.

If M is a connected topological manifold and m a fixed point in Int M, then $H(M)$ will denote the topological group of homeomorphisms of M under the compact-open topology, and $H(M, m)$ the closed subgroup of homeomorphisms which leave m fixed.

The projection of $M\times[0, 1]$ onto M will be denoted by pr_M. If H is a weak isotopy between two homeomorphisms of (M, m) then the curve

$$\gamma\colon [0, 1] \to M,$$

defined by $\gamma(t) = pr_M(H(m, t))$, is a closed curve in M based at m, which we call the *trace* of H.

2. Restriction of isotopies.

THEOREM 2.1. *Let M be a connected and simply connected topological manifold and m a fixed point in* Int M. *If h_0 and h_1 are two isotopic homeomorphisms of M, each of which leaves m fixed, then $h_0/M-m$ and $h_1/M-m$ are isotopic homeomorphisms of $M-m$.*

Let

$$H\colon M \times [0, 1] \to M \times [0, 1]$$

be an isotopy between h_0 and h_1, and let

$$h_t\colon M \to M$$

be the homeomorphism of M defined by

$$H(x, t) = (h_t(x), t).$$

The following facts are well known.

(i) The map $\Gamma\colon [0, 1]\to H(M)$, defined by $\Gamma(t) = h_t$, is continuous.

(ii) $H(M)$ is a principal bundle over Int M with fibre and group $H(M, m)$ and projection $p\colon H(M)\to \text{Int } M$ defined by $p(h) = h(m)$.

Then $\gamma = p\Gamma$, the trace of the isotopy H, is contractible because M is simply connected. Hence by the covering homotopy theorem, Γ can be deformed into a path Γ' which connects h_0 with h_1 and lies entirely in the fibre $p^{-1}(m) = H(M, m)$. Then

$$H'\colon M \times [0, 1] \to M \times [0, 1],$$

defined by

$$H'(x, t) = (\Gamma'(t)(x), t),$$

is an isotopy between h_0 and h_1 such that, for all $t \in [0, 1]$,

$$H'(m, t) = (m, t).$$

Hence $H'/(M-m)\times[0, 1]$ is an isotopy between $h_0/M-m$ and $h_1/M-m$.

3. **Homma's theorem.** Homma [2] has recently proved the following theorem, in the statement of which, $U_\epsilon(\tilde{P}^k)$ denotes the set of points whose distance from $\tilde{P}^k$ is less than ϵ.

HOMMA'S THEOREM. *Let M^n, $\tilde{M}^n$ and $\tilde{P}^k$ be two finite combinatorial n-manifolds and a finite polyhedron such that $\tilde{M}^n$ is topologically embedded in M^n, $\tilde{P}^k$ is piecewise linearly embedded in* Int $\tilde{M}^n$ *and $2k+2 \leqq n$. Then for any $\epsilon > 0$, there is an ϵ-homeomorphism F of M^n onto M^n such that*

$$F/M^n - U_\epsilon(\tilde{P}^k) = 1,$$

$$F/\tilde{P}^k \text{ is piecewise linear.}$$

Combining the reciprocal approximation technique employed by Homma to prove the above theorem with Lemma 2 of [2], one easily obtains the following result, which may be regarded as an indirect corollary to Homma's theorem.

THEOREM 3.1. *Let M^n be a topological n-manifold with boundary B^{n-1}. Let M_1^n and M_2^n be two combinatorial n-manifolds, each of which has M^n for underlying space. Let P_1 be a polygonal arc in M_1^n which meets B_1^{n-1} only at its endpoints. If $n \geqq 4$, then for any $\epsilon > 0$ there is an ϵ-homeomorphism F: $M_1^n \to M_2^n$ such that*

$$F/M_1^n - U_\epsilon(P_1) = 1,$$

$$F/B_1^{n-1} = 1,$$

F/P_1 is piecewise linear.

4. **Restriction of weak isotopies.**

THEOREM 4.1. *Let M be a connected and simply connected topological manifold which can support a piecewise linear structure, and m a fixed point in* Int M. *If h_0 and h_1 are two weakly isotopic homeomorphisms of M, each of which leaves m fixed, then $h_0/M-m$ and $h_1/M-m$ are weakly isotopic homeomorphisms of $M-m$.*

Since M can support a piecewise linear structure, triangulate $M\times[0, 1]$ as a combinatorial manifold in which $m\times[0, 1]$ appears as a subcomplex. Let

$$H: M \times [0, 1] \to M \times [0, 1]$$

be a weak isotopy between h_0 and h_1. The plan is to first find a homeomorphism F of $M\times[0, 1]$ onto itself such that

$$F/(M \times 0) \cup (M \times 1) = 1,$$
$$FH(m\times[0, 1]) \text{ is polygonal},$$

and then a homeomorphism F' of $M\times[0, 1]$ onto itself such that

$$F'/(M \times 0) \cup (M \times 1) = 1,$$
$$F'FH(m \times [0, 1]) = m \times [0, 1].$$

Then $F'FH$ will be a weak isotopy of h_0 with h_1 which takes $m\times[0, 1]$ onto itself, and hence $F'FH/(M-m)\times[0, 1]$ will be a weak isotopy of $h_0/M-m$ with $h_1/M-m$.

If dim $M=1$, M is homeomorphic to an open, half-closed or closed arc, and the theorem is trivially true.

If dim $M=2$, suppose first that M is homeomorphic to S^2. The existence of both F and F' is demonstrated in §9 of [**3**]. If M is not homeomorphic to S^2, then Int M is homeomorphic to Euclidean 2-space, R^2. The existence of F is shown in §9 of [**3**], while the existence of F' follows from a standard argument involving Dehn's lemma [**4**] and the fact that an orientation preserving homeomorphism of a 2-sphere is isotopic to the identity.

If dim $M\geqq 3$, let M_2^n be $M\times[0, 1]$ triangulated as above, and let M_1^n be $M\times[0, 1]$ with the triangulation induced from M_2^n by the homeomorphism H. Since $m\times[0, 1]$ appears as a subcomplex of M_2^n, $H(m \times [0, 1])$ appears as a subcomplex of M_1^n. Letting $P_1 =H(m\times[0, 1])$, the existence of F is assured by Theorem 3.1.

Since $M \times [0, 1]$ is simply connected, the polygonal arc $FH(m\times[0, 1])$ is homotopic to the polygonal arc $m\times[0, 1]$ in $M\times[0, 1]$. Since dim $(M\times[0, 1])\geqq 4$, a general position argument will produce F'.

5. **An application.** Think of S^n as the one-point compactification of R^n by the point ∞. Then the following may be regarded as a corollary to Theorem 4.1.

THEOREM 5.1. *If h is a homeomorphism of (S^n, ∞) which is weakly isotopic to the identity, then h/R^n is weakly isotopic to the identity homeomorphism of R^n.*

For the theorem is trivial when $n=1$ and S^n is simply connected when $n>1$.

Now let h be a homeomorphism of (S^n, ∞), and from $S^n \times [0, 1]$ form a space M by identifying $(x, 0)$ with $(h(x), 1)$ for each $x \in S^n$. Let $\phi: S^n \times [0, 1] \to M$ be the decomposition map.

THEOREM 5.2. *If M is homeomorphic to $S^n \times S^1$, then $\phi(R^n \times [0, 1])$ is homeomorphic to $R^n \times S^1$.*

If M is homeomorphic to $S^n \times S^1$, then it follows from [5] that h must be weakly isotopic to the identity. By the preceding theorem, h/R^n must also be weakly isotopic to the identity, from which it easily follows that $\phi(R^n \times [0, 1])$ is homeomorphic to $R^n \times S^1$.

6. **Further results.** Theorem 2.1 is actually a special case of a more general result, which is briefly described below.

Let M be a connected manifold and $m \in \text{Int } M$. Let h be a homeomorphism of M leaving m fixed, which is isotopic to the identity homeomorphism, 1_M. Define the *trace class*, $\tau(h)$, to be the set of all elements of $\pi_1(M, m)$ which can be represented by traces of isotopies of 1_M with h. Then $\tau(1_M)$ is a central (and hence normal) subgroup of $\pi_1(M, m)$, and $\tau(h)$ is a coset of $\tau(1_M)$. Thus $\tau(h)$ may also be regarded as an element of the *trace group*

$$T(M, m) = \pi_1(M, m)/\tau(1_M).$$

Now, if h_0 and h_1 are isotopic homeomorphisms of M, each of which leaves m fixed, then $h_0^{-1}h_1$ is isotopic to 1_M, hence $\tau(h_0^{-1}h_1)$ is defined. It then follows easily from the covering homotopy theorem applied to the bundle $H(M)$ over Int M that $h_0/M-m$ and $h_1/M-m$ are isotopic homeomorphisms of $M-m$ if and only if $\tau(h_0^{-1}h_1) = \tau(1_M)$.

This condition is automatically satisfied when M is simply connected, hence Theorem 2.1.

Theorem 4.1 follows from a similar result about weak isotopy.

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