RESEARCH ANNOUNCEMENTS

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ANALYTIC MEASURES ON COMPACT GROUPS

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The purpose of this note is the announcement of an extension to compact abelian groups of the two celebrated theorems of F. and M. Riesz [8] concerning analytic measures on the circle group. The content of these theorems is as follows:

Let μ be a Borel measure on the circle satisfying

$$\int_{-\pi}^{+\pi} e^{in\theta} d\mu(\theta) = 0, \qquad n = 1, 2, 3, \cdots.$$

Then

A. *μ is absolutely continuous with respect to Lebesgue measure and*

B. *If μ vanishes identically*[2] *on a set of positive Lebesgue measure, then μ must be the zero measure.*

It is not hard to see that A and B together are equivalent to the following:

The collection of Borel sets on which μ vanishes identically is invariant under rotation.

This is the assertion concerning analytic measures that we extend to compact groups. We also shall state several of its consequences, including analogues of A and B. The work was inspired by, and is in part an extension of, several of the results of Helson and Lowdenslager [4; 5].

In all that follows G is a compact abelian group,[3] $\hat{G}$ its discrete dual, and ψ is a fixed homomorphism of $\hat{G}$ into the group R of real numbers. The mapping $\psi: \hat{G} \to R$ is a continuous homomorphism and

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[2] We say that a measure μ *vanishes identically* on a set E if μ vanishes on all Borel subsets of E.

[3] See however our final remarks.

thus induces a continuous homomorphism $\phi: R \to G$ of the associated dual groups; ϕ is the unique mapping of R into G satisfying

$$\sigma(\phi(t)) = e^{i\psi(\sigma)t}, \qquad t \in R, \ \sigma \in \hat{G}.$$

We shall use $\hat{}$ to denote Fourier transform, $*$ to denote convolution and by *measure* we shall mean finite complex regular Borel measure. If μ is a measure, $|\mu|$ is the associated total variation measure.

A function or measure on G is called *ϕ-analytic* if its Fourier transform vanishes on $\{\sigma: \sigma \in \hat{G}, \ \psi(\sigma) < 0\}$. A measure μ on G is called *quasi-invariant under* ϕ if $\{E: E \text{ Borel}, \ |\mu|(E) = 0\}$ is invariant under translation by the elements of $\phi(R)$.

MAIN THEOREM. *Let μ be a ϕ-analytic measure. Then μ is quasi-invariant under ϕ.*

Denote by ρ the image of the measure $(1/1+x^2)dx$ on R under the mapping $\phi: R \to G$. It is not hard to show that a measure μ on G is quasi-invariant under ϕ if and only if $|\mu|$ and $\rho * |\mu|$ are mutually absolutely continuous. Thus we have a reformulation.

MAIN THEOREM. *Let μ be a ϕ-analytic measure on G. Then $|\mu|$ and $\rho * |\mu|$ are mutually absolutely continuous.*

Before stating the first consequences of this result some further definitions are necessary. If E is a Borel subset of G we shall say that E is of *measure zero in the direction of* ϕ of each coset $x + \phi(R)$ intersects E in a set of linear measure zero; more precisely, if for each x in G,

$$\{t: t \in R, \ x + \phi(t) \in E\}$$

has Lebesgue measure zero. A measure μ on G that vanishes on each subset of G which is of measure 0 in the direction of ϕ is called *absolutely continuous in the direction of* ϕ. It can be shown that μ is absolutely continuous in the direction of ϕ if and only if it *translates continuously in the direction of* ϕ; that is, if

$$\lim_{t \to 0} \|\mu_t - \mu\| = 0,$$

where $\|\cdot\|$ is the total variation norm, and for each t in R the translated measure μ_t is defined by

$$\mu_t(E) = \mu(\phi(t) + E), \qquad E \text{ Borel}.$$

(For the circle group this result is due to Plessner [7].)

A measure quasi-invariant under ϕ is easily shown to be absolutely continuous in the direction of ϕ, so by the Main Theorem we have the following analogue of assertion A above.

THEOREM A. *Let μ be a ϕ-analytic measure on G. Then μ is absolutely continuous in the direction of ϕ.*

For μ a measure on G, the *ψ-conjugate* of μ is defined to be that measure μ_ψ (if such exists) whose Fourier transform satisfies

$$\hat{\mu}_\psi(\sigma) = \begin{cases} \hat{\mu}(\sigma), & \psi(\sigma) > 0 \\ 0, & \psi(\sigma) = 0 \\ -\hat{\mu}(\sigma), & \psi(\sigma) < 0. \end{cases}$$

Theorem A is equivalent to the assertion that each measure on G having a ψ-conjugate is absolutely continuous in the direction of ϕ.

Theorem A together with the result of Bishop [1] yields the following, which for the circle group is due to Rudin [9] and Carleson [3].

COROLLARY 1. *Let E be a closed subset of G. Then the following are equivalent:*

1°. *E is of measure zero in the direction of ϕ.*

2°. *For each continuous function g on E there is a continuous ϕ-analytic function f on G that agrees on E with g.*

If H is the n-torus, its dual $\hat{H}$ is the group of lattice points in real n-space. Bochner's extension of the F. and M. Riesz Theorem (see [2]) states that any measure on the n-torus whose Fourier transform vanishes off the positive octant of the lattice points must be absolutely continuous. Theorem A applied n-times yields the following, which includes the Bochner Theorem.

COROLLARY 2. *Let H be the n-torus, μ a measure on H and F a set of n homomorphisms of $\hat{H}$ into R. Assume that the set F is linearly independent and that for each ψ in F the conjugate measure μ_ψ exists. Then μ must be absolutely continuous.*

One further definition is necessary before we can state our extension of assertion B above. For E a Baire subset of G we denote by E_ϕ the union of all cosets $x+\phi(R)$ that intersect E in a set of positive linear measure. More precisely, E_ϕ consists of those x in G for which

$$\{t: t \in R,\ x + \phi(t) \in E\}$$

has positive Lebesgue measure.

THEOREM B. *Let μ be a ϕ-analytic measure on G. Suppose that E is a Baire subset of G on which μ vanishes identically. Then μ vanishes identically on E_ϕ.*

As a special case we have the following result, the second half of which is due to Helson, Lowdenslager and Malliavin [5].

COROLLARY 3. *Assume that $\phi(R)$ is dense in G. Let μ be a ϕ-analytic measure on G that either*

(1) *vanishes identically on an open subset of G*

or

(2) *is absolutely continuous with respect to Haar measure and vanishes identically on a Borel set of positive Haar measure.*

Then μ is the zero measure.

The next result is a simple consequence of the Main Theorem. A special case of the proposition has also been obtained by Frank Forelli using quite different methods. The corollaries that we list are in part refinements of results of Helson-Lowdenslager [4] and Bochner [2]. Bochner has informed us that he has been able to obtain the corollaries using the results of [4].

PROPOSITION. *Let μ be a ϕ-analytic measure on G (or more generally, any measure on G quasi-invariant under ϕ). Let η be a measure on G that is the image of some measure on R under the mapping $\phi: R \to G$. Then the convolution $\eta * \mu$ is absolutely continuous with respect to μ. In particular, if μ is singular with respect to Haar measure, $\eta * \mu$ is either singular with respect to Haar measure or is the zero measure.*

To simplify the statements of the corollaries we assume that $\hat{G}$ is R with the discrete topology and $\psi: \hat{G} \to R$ the identity mapping, so that G is the Bohr compactification of the reals.

COROLLARY 4. *Let K be a closed subset of R. Let μ be a ϕ-analytic measure on G, λ its singular part. If $\hat{\mu}$ vanishes off K then $\hat{\lambda}$ also vanishes off K.*

For $K = \{t: t \leqq 0\}$, this is due to Helson-Lowdenslager [4].

COROLLARY 5. *Let μ be a singular ϕ-analytic measure on G. Then $\{\sigma: \hat{\mu}(\sigma) \neq 0\}$ is a subset of R containing no isolated points.*

COROLLARY 6. *Let K be a countable closed subset of R. Let μ be a ϕ-analytic measure on G whose Fourier transform vanishes off K. Then μ is absolutely continuous with respect to Haar measure.*

There are several questions connected with the above results which deserve mention. First, most of our deductions from the Main Theorem are valid in the context of one-parameter groups of homeomorphisms of compact topological spaces. It is conceivable that a version of the Main Theorem itself is also valid in this context. Here is a possible generalization. Let X be a compact space and $\{T_t\}$ a one-parameter group of homeomorphisms of X. Call a measure μ on X $\{T_t\}$-*analytic* if the vector valued integral

$$\int_{-\infty}^{+\infty} h(t)T_t\mu dt$$

is zero for all h in $L^1(R)$ whose Fourier transforms vanish for $t \leqq 0$. (In the case that $X = G$ and T_t is translation by $\phi(t)$, this agrees with our previous definition of analyticity.) Then a generalization of our Main Theorem would be the assertion that a $\{T_t\}$-analytic measure μ is quasi-invariant under $\{T_t\}$; that is, the collection of $|\mu|$-null sets is $\{T_t\}$ invariant. Indeed, with this definition of analyticity (and T_t translation by $\phi(t)$), the Main Theorem continues to hold even when G is noncommutative.

Another possible extension of some of our results is to the context of Dirichlet algebras (for the relevant definitions, see [6]). The collection of ϕ-analytic continuous functions on G is a Dirichlet algebra on G. Theorem A says precisely that each Borel subset of G that is of measure zero for all of the *representing measures* for the algebra must be of measure zero for all of the *annihilating measures* for the algebra. It is conceivable that a corresponding result holds for a wider class of Dirichlet algebras.

BIBLIOGRAPHY

1. E. Bishop, *A general Rudin-Carleson theorem*, Proc. Amer. Math. Soc. **13** (1962), 140–143.

2. S. Bochner, *Boundary values of analytic functions in several variables and of almost periodic functions*, Ann. of Math. **45** (1944), 708–722.

3. L. Carleson, *Representations of continuous functions*, Math. Z. **66** (1957), 447–451.

4. H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. **99** (1958), 165–201.

5. ———, *Prediction theory and Fourier series in several variables*. II, Acta Math. **106** (1961), 175–212.

6. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

7. I. Plessner, *Eine Kennzeichnung der totalstetig Funktionen*, J. Reine Angew. Math. **160** (1929), 26–32.

8. F. Riesz and M. Riesz, *Über die Randwerte einer analytischen Function*, Quatrième Congrès des Math. Scand., 1916, pp. 27–44.

9. W. Rudin, *Boundary values of continuous analytic functions*, Proc. Amer. Math. Soc. **7** (1956), 808–811.

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