SOLUTIONS OF EQUATIONS OVER GROUPS

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A polynomial equation of degree n over a field K can always be solved in a suitable extension K' of degree at most n over K. One might expect to find a corresponding result for groups; it is the purpose of this note to show that this is, in fact, the case.

The analogous problem for groups is to find a *solution* for the *equation*

$$x^{n_1}g_1x^{n_2}g_2 \cdots x^{n_k}g_k = 1 \tag{1}$$

in the *unknown* x, where the g_i are elements of a group G and all the n_i are non-negative, i.e., to find an element g' in a group G' in which G is embedded such that

$$g'^{n_1}g_1g'^{n_2}g_2 \cdots g'^{n_k}g_k = 1$$

in G'. It will be shown that (1) can always be solved over any group G, with a solution in G', where G' is, in some sense, an extension group of "degree n" over G, for $n = \sum_{i=1}^{k} n_i$. This result (coupled with the fact that the equation

$$x^{-1}g_1xg_2 = 1$$

is solvable over any torsion-free group [2]) gives hope for the conjecture that any equation is solvable over a torsion-free group.

The solution we will give for (1) uses the construction used by Baumslag [1] to solve the equation $x^ng = 1$ (cf. [3]).

Without loss of generality, we may restrict our attention to equations of the form

$$xa_0xa_1 \cdots xa_{n-1} = 1, \qquad a_i \in G, \tag{2}$$

where some of the a_i are possibly 1. The solution of (2) will be constructed in the wreath product of G and a cyclic group C of order n. To fix notation, we will outline the definition of the wreath product, GWrC (cf. [2]): Let G^C be the group of all mappings $\{f\}$ of C into G with $ff' \in G^C$ defined by $ff'(t) = f(t)f'(t)$, for all $t \in C$. GWrC is the group composed of the set $\{sf \mid s \in C, f \in G^C\}$ with

$$sf \cdot s'f' = ss'f^{s'}f',$$

where $f^{s'}(t) = f(ts'^{-1})$ for all $t \in C$. We will embed G in GWrC by identifying $1'g^o \in$ GWrC with $g \in G$, where $g^o(t) = g$ for all $t \in C$ and $1'$ is the neutral element of C.

The main result of this note is stated as follows:

THEOREM 1. *Let G be an arbitrary group, $C=gp(c)$ be a cyclic group of order n. A solution of equation* (2) *is given by $c^{-1}f\in$*GWrC, *where $f(c^i)=a_i^{-1}$, $i=0, 1, \cdots, n-1$.*

In other words, Theorem 1 states that

$$(3) \qquad (c^{-1}f)\cdot(1'\overset{o}{a}_0)\cdot(c^{-1}f)\cdot(1'\overset{o}{a}_1)\cdots(c^{-1}f)\cdot(1'\overset{o}{a}_{n-1}) = 1'\overset{o}{1},$$

(1 is the neutral element of G), which can be verified by a straightforward application of the definition of GWrC.

There are several properties of a group G which are inherited by GWrC. For instance: if G is finite, GWrC is finite; if G has finite exponent m, GWrC has exponent mn; if G is soluble, GWrC is soluble of length at most one greater than G.

A group which has the property that every equation of type (2) has a solution in the group itself will be called a *P-algebraically closed group*. Such a group is also divisible, i.e., contains a solution for every equation $x^n g=1$. Some of the results in [1] can be extended immediately to P-algebraically closed groups. In particular, Corollary 4.3 and Theorem 4.4 of [1], respectively, have the following extensions:

COROLLARY 1 (*cf., also, Neumann* [3]). *Every group G can be embedded in a P-algebraically closed group.*

THEOREM 2. *Every periodic group can be embedded in a periodic P-algebraically closed group.*

For the solution of (2) it is sufficient, of course, to consider the subgroup W' of GWrC generated by G and $c^{-1}f$ rather than the whole of GWrC. W' has, in fact, the same solubility length as G; this result can be checked rather easily here. Consequently, we can state

THEOREM 3. *If G is soluble of length q, a solution of equation* (2) *can be found in an overgroup W' of G which is soluble of length q.*

REFERENCES

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