ON A THEOREM OF G. W. MACKEY

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Let G be a locally compact group, H a closed subgroup of G, and L a unitary representation of H on the Hilbert space $\mathcal{V}$. Let U^L be the representation of G induced by L (see [1] for the definitions and notations of the present note). For $h \in C_0(G/H)$ and $f \in \mathcal{K}(U^L)$, set $(E^L(h)f)(x) = h(\pi(x))f(x)$, where π is the natural projection of G on G/H. Then E^L is a *-homomorphism of the commutative *-algebra $C_0(G/H)$ (under the pointwise operations) into $\mathcal{L}(\mathcal{K}(U^L), \mathcal{K}(U^L))$. Let M be another unitary representation of H on a Hilbert space $\mathcal{W}$ and form U^M and E^M. If $T \in \mathcal{R}(L, M)$, the space of operators intertwining L and M, define $T^\sim$ by setting $(T^\sim f)(x) = Tf(x)$, $f \in \mathcal{K}(U^L)$. Clearly, $T^\sim \in \mathcal{R}(U^L, U^M) \cap \mathcal{R}(E^L, E^M)$. $T \to T^\sim$ is functorial in the sense that it commutes with operator multiplication and adjunction and sends any identity (resp. zero) operator onto an identity (resp. zero) operator.

The following theorem is due to G. W. Mackey in the case that G, $\mathcal{V}$, and $\mathcal{W}$ are separable [2, Theorem 6.4]. His techniques are measure theoretic and do not seem suited to the nonseparable case.

THEOREM. $T \to T^\sim$ *is an isomorphism onto.*

PROOF. (a) *Linearity is clear.* Suppose $T_1^\sim = T_2^\sim$. For $\phi \in C_0(G)$, $v \in \mathcal{V}$, form

$$\epsilon_L(\phi, v)(x) = \int_H \phi(\xi x)\delta(\xi)^{-1/2}\Delta(\xi)^{1/2}L_\xi^{-1}v \, d\xi,$$

as in [1]. (All integrations in this paper are with respect to right Haar measure. Δ (resp. δ) is the modular function of G (resp. H).) $\epsilon_L(\phi, v) \in \mathcal{K}(U^L)$. $\epsilon_M(\phi, w)$, $w \in \mathcal{W}$ defined similarly, $\in \mathcal{K}(U^M)$. Moreover, $T_i^\sim \epsilon_L(\phi, v) = \epsilon_M(\phi, T_i v)$. Therefore $\epsilon_M(\phi, T_1 v)(e) = \epsilon_M(\phi, T_2 v)(e)$ for all $\phi \in C_0(G)$. Letting $\phi | H$ approach the δ-measure, we obtain $T_1 v = T_2 v$, $v \in \mathcal{V}$. Therefore $T \to T^\sim$ is one-to-one.

(b) *Suppose $L = M$.* To show the mapping is onto, it is sufficient to show that if $T_0 \in \mathcal{R}(U^L, U^L) \cap \mathcal{R}(E^L, E^L)$, $0 \leq T_0 \leq I$, then $T_0 = T^\sim$ for some $T \in \mathcal{R}(L, L)$.

We define a linear map S of $C_0(G) \otimes \mathcal{K}(U^L)$ into $\mathcal{V}$ by setting $S(\phi \otimes f) = \int_G \phi(y)f(y^{-1})dy$. Now $S(\phi \otimes \epsilon(\psi, v)) = \epsilon(\psi*\phi, v)(e)$. Therefore

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$S(\phi \otimes \epsilon(\psi, v)) \to v$ as $\psi * \phi | H \to$ the δ-measure, and the range of S is dense in $\mathcal{V}$. Again,

$$(S(\phi \otimes f), S(\psi \otimes g))$$

$$= \int \int \phi(y) [\psi(x)]^- (f(y^{-1}), g(x^{-1})) dx \, dy$$

$$= \int \int \phi(yx^{-1}) [\psi(x^{-1})]^- \Delta(x)^{-1} ((U_y^L \!{}^{-1} f)(x), g(x)) dx \, dy$$

$$= \int (E^L(\{\phi, \psi\}(\cdot, y)) U_y^L \!{}^{-1} f, g) dy,$$

where $\{\phi, \psi\}$ is the function in $C_0(G/H \times G)$ defined by

$$\{\phi, \psi\}(\pi(x), y) = \int_H \phi(y(\xi x)^{-1}) [\psi((\xi x)^{-1})]^- \Delta(\xi x)^{-1} d\xi.$$

This follows from the fact that if $f, g \in \mathfrak{IC}(U^L)$ and if $h \in C_0(G)$, then $(E^L(\tau h) f, g) = \int h(x)(f(x), g(x)) dx$, where $(\tau h)(\pi(x)) = \int_H f(\xi x) d\xi$. For any $\lambda \in C_0(G/H \times G)$, we define $\Phi(\lambda) \in \mathcal{L}(\mathfrak{IC}(U^L), \mathfrak{IC}(U^L))$ by setting

$$\Phi(\lambda) = \int E^L(\lambda(\cdot, y)) U_y^L \!{}^{-1} dy.$$

We have shown $(S(\phi \otimes f), S(\psi \otimes g)) = (\Phi(\{\phi, \psi\}) f, g)$.
    Now

$$\left\| S\left(\sum \phi_i \otimes T_0^{1/2} f_i\right) \right\|^2 = \sum_{ij} (\Phi(\{\phi_i, \phi_j\}) T_0^{1/2} f_i, T_0^{1/2} f_j)$$

$$= \sum_{ij} (T_0 \Phi(\{\phi_i, \phi_j\}) f_i, f_j),$$

because $\Phi(\lambda)$ commutes with $T_0^{1/2}$, $\lambda \in C_0(G/H \times G)$. Adding this equation to the equation got from it by replacing T_0 with $I - T_0$, we obtain

$$\left\| S\left(\sum \phi_i \otimes T_0^{1/2} f_i\right) \right\|^2 \leq \left\| S\left(\sum \phi_i \otimes T_0^{1/2} f_i\right) \right\|^2 + \left\| S\left(\sum \phi_i \otimes (I - T_0)^{1/2} f_i\right) \right\|^2$$

$$= \sum_{ij} (\Phi(\{\phi_i, \phi_j\}) f_i, f_j) = \left\| S \sum \phi_i (\otimes f_i) \right\|^2.$$

Since $\sum \phi_i \otimes f_i \to \sum \phi_i \otimes T_0^{1/2} f_i$ is linear, there is a unique bounded linear operator T_1 on $\mathcal{V}$ such that $T_1 S(\phi \otimes f) = S(\phi \otimes T_0^{1/2} f)$, all ϕ and f. But $L_\xi S(\phi \otimes f) = \delta(\xi)^{-1/2} \Delta(\xi)^{1/2} S((R_\xi \phi) \otimes f)$, all $\xi \in H$, where $(R_\xi \phi)(x) = \phi(x\xi)$. It follows that $T_1 \in \mathfrak{R}(L, L)$.
    Let $T = T_1^* T_1 \in \mathfrak{R}(L, L)$. We have

$$(S(\phi \otimes T^\sim f), S(\psi \otimes g))$$
$$= (TS(\phi \otimes f), S(\psi \otimes g)) = (S(\phi \otimes T_0^{1/2}f), S(\psi \otimes T_0^{1/2}g))$$
$$= (\Phi(\{\phi, \psi\})T_0^{1/2}f, T_0^{1/2}g) = (\Phi(\{\phi, \psi\})T_0 f, g)$$
$$= (S(\phi \otimes T_0 f), S(\psi \otimes g)).$$

Hence $\int \phi(y)(T^\sim f)(y^{-1})dy = \int \phi(y)(T_0 f)(y^{-1})dy$ for all $\phi \in C_0(G)$. It follows that $T^\sim f = T_0 f$, all f, and $T_0 = T^\sim$.

(c) *General case.* Let $T_0 \in \mathfrak{R}(U^L, U^M) \cap \mathfrak{R}(E^L, E^M)$. Now $U^{L \oplus M} \cong U^L \oplus U^M$ in a canonical way. Using this isomorphism and matrix notation, set

$$T_1 = \begin{pmatrix} 0 & 0 \\ T_0 & 0 \end{pmatrix}.$$

$T_1 \in \mathfrak{R}(U^{L \oplus M}, U^{L \oplus M}) \cap \mathfrak{R}(E^{L \oplus M}, E^{L \oplus M})$. Thus $T_1 = T_2^\sim$ where $T_2 \in \mathfrak{R}(L \oplus M, L \oplus M)$. Write

$$T_2 = \begin{pmatrix} * & * \\ T & * \end{pmatrix}.$$

$T \in \mathfrak{R}(L, M)$. It is easily seen that $T_0 = T^\sim$.

REMARK. It follows immediately from this theorem that (U^L, E^L) is equivalent to (U^M, E^M) if and only if $L \cong M$ and that (U^L, E^L) is irreducible if and only if L is irreducible.

REFERENCES

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