RESOLUTION OF POSITIVE OPERATORS

BY R. V. CHACON[1]

We begin with a σ-finite measure space $(S, \mathfrak{F}, \mu)$ and a positive linear operator T on L_1 with $\|T\|_1 \leq 1$. The main result of this note is the following theorem:

THEOREM 1. *If $p \in L_1$ and is non-negative and if T is regular, then*

$$L_1 \overset{p}{=} I_p \oplus \overline{M}{}^2$$

where

$$M = \{g : g = f - Tf, f \in L_1\},$$
$$I_p = \{hp : T(hp) = hTp, hp \in L_1\}.$$

Before proceeding we give some definitions, in particular that of a *regular operator*. We have used this term in the statement of Theorem 1.

DEFINITION. *T is regular if*

$$\sum_{k=0}^{\infty} T^k p(s) = +\infty$$

whenever $p \in L_1$ is strictly positive.

We summarize results [2], reducing general operators to regular operators in the following theorem:

THEOREM 2. *There exists a set S_1 so that the restriction of T to $L_1(S_1, \mathfrak{F}_1, \mu)$, where $\mathfrak{F}_1 = \mathfrak{F} \cap S_1$, is regular and so that $\sum_{k=0}^{\infty} |T^k f| < +\infty$ on $c\, S_1$. Furthermore, for each $f \in L_1(S_1, \mathfrak{F}_1, \mu)$ there exists a $f_{S_1} \in L_1(S_1, \mathfrak{F}_1, \mu)$ so that*

$$\lim_{n \to \infty} \frac{\displaystyle\sum_{k=0}^{n} T^k(f - f_{S_1})}{\displaystyle\sum_{k=0}^{n} T^k p} = 0$$

on $S_1 \cap \{s : \sum_{k=0}^{\infty} T^k p(s) > 0\}$.

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[2] $A \overset{p}{=} B$ if the spaces are equal with respect to the seminorm obtained by integration over the support of $\sum_{k=0}^{\infty} T^k p(s)$.

Since L_1^* is isometric to L_∞, we regard T^*, the adjoint of T, as a linear operator on L_∞. It follows that T^* is positive and that $\| T^* \|_\infty \leqq 1$.

DEFINITION. *Let I_∞ be the space of functions of L_∞ which are invariant under T^*, let $\mathfrak{g}$ be the Borel field generated by the functions of I_∞ and let I_M be the class of functions which are measurable with respect to $\mathfrak{g}$ (I_M is not assumed to be a subspace of L_∞ or L_1).*

THEOREM 3. *If $p \in L_1$ and is non-negative and if T is regular, then*

$$\{ hp \colon T(hp) = hTp, \, hp \in L_1 \} = \{ hp \colon hp \in L_1, \, h \in I_M \}.$$

Theorems 1 and 3 can be combined to yield the following result:

THEOREM 4. *If $p \in L_1$ and is non-negative and if T is regular, then*

$$L_1 \overset{p}{=} I_p \oplus \overline{M}$$

where

$$M = \{ g \colon g = f - Tf, f \in L_1 \},$$
$$I_p = \{ hp \colon hp \in L_1 \text{ and } T^*h = h \}$$

and $T^*h = h$ provided the domain of definition of T^* is extended in a natural way.

We note that this result is analogous to the Kakutani-Yosida mean ergodic theorem in which a similar resolution is obtained for operators on *reflexive* Banach spaces. This analogy is more apparent if the reflexive Banach space is Hilbert space and we state their theorem specialized to the Von Neumann case:

MEAN ERGODIC THEOREM. *If H is Hilbert space and if T is a linear transformation on H with $\| T \| \leqq 1$ then*

$$H = I \oplus \overline{M}$$

where

$$M = \{ g \colon g = f - Tf, f \in H \},$$
$$I = \{ f \colon T^*f = f \}.$$

Since the fixed points of a contraction in Hilbert space are the same as those of its adjoint, $I = \{ f \colon Tf = f \}$.

Theorem 1 can be used to obtain the results of [1] and [3], avoiding the somewhat complicated procedure if [1] which involves showing that the integral of a function is infinite if the limit of the ratio in question fails to exist. We have, in a forthcoming paper, also obtained some

results for operators not assumed positive by means of a reduction procedure [4] and the present theorems.

We are indebted to the work of Eberhard Hopf for suggesting that a resolution of this type is possible.

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TCHEBYCHEFF QUADRATURE IS POSSIBLE ON THE INFINITE INTERVAL[1]

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The purpose of this announcement is to state a theorem on Tchebycheff quadrature which answers a question posed in [1], and to discuss the proof. Complete details will appear elsewhere.

1. Tchebycheff quadrature.

DEFINITION 1.1. A unit mass distribution on $(-\infty, \infty)$ possessing moments of all positive integer order will be said to belong to class D.

DEFINITION 1.2. Let ψ be an element of D and n a positive integer. We refer to the equations

$$\frac{1}{n} \sum_{i=1}^{n} x_{i,n}^{k} = \int x^{k} d\psi, \qquad k = 1, \cdots, n$$

as the equations (ψ, n). These equations admit a solution $x_{1,n}, \cdots, x_{n,n}$ which is unique up to permutation of the first index.

DEFINITION 1.3. T quadrature is said to be possible for an element ψ of D if equations (ψ, n) have real solutions for every positive integer n. If T quadrature is possible for ψ it is called a T distribution.