ON THE EXTENSION PROPERTY FOR COMPACT OPERATORS[1]

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Communicated by M. M. Day, June 4, 1962

1. **Introduction.** The main theorem of this note shows the equivalence of several extension properties for operators and contains also some geometrical characterizations of the spaces having these properties. In particular a characterization of such spaces is given in terms of intersection properties of their cells, similar to that given by Nachbin for $\mathcal{P}_1$ spaces [**8**]. Our theorem extends previous results of Grothendieck [**4**]. Some applications are given, among them a new characterization of $C(K)$ spaces. In this connection some problems raised by Aronszajn and Panitchpakdi [**1**], Grothendieck [**4**] and Nachbin [**8**; **9**] are solved.

Notations. All Banach spaces are assumed to be over the reals. $S_X(x_0, r)$ denotes the cell $\{x; x\in X, \|x-x_0\| \leqq r\}$. A Banach space X is called a $\mathcal{P}_\lambda$ space if from any Z containing X there is a projection on X with norm $\leqq\lambda$ (see Day [**2**, pp. 94–96]). We say that a Banach space has the metric approximation property (M.A.P.) if for every compact subset K of X and every $\epsilon>0$ there is a compact operator T from X into itself such that $\|T\|=1$ and $\|Tx-x\|\leqq\epsilon$ for $x\in K$. A (possibly) stronger property was introduced by Grothendieck [**3**, pp. 187–191]. He proved that the common Banach spaces have the M.A.P. It is an open problem whether there exists a Banach space which does not have the M.A.P.

2. **The main results.** We state now the main result of this note (the equivalencies $1\leftrightarrow 2\leftrightarrow 3\leftrightarrow 5$ are due to Grothendieck [**4**]). In the extension properties stated below Y, Z and V will be arbitrary Banach spaces satisfying $Z\supset Y$, $V\supset X$ and the indicated restrictions (if any).

THEOREM 1. *Let X be a Banach space; then the following statements are equivalent:*

[1] The research reported in this note has been sponsored in part by the Air Force Office of Scientific Research of the Air Research and Development Command U. S. Air Force through its European Office. It is a modified version of Technical Note No. 22 (Jerusalem, October 1961).

[2] This note is based on a part of a Ph.D. thesis which was written under the supervision of Professor A. Dvoretzky and Dr. B. Grünbaum. I wish to express my warm thanks to both for their valuable help and kind encouragement.

(1) *X^{**} is a $\mathcal{P}_1$ space.*

(2) *X^* is an $L_1(\mu)$ space (for some measure μ).*

(3) *Every compact operator T from Y to X has, for every $\epsilon>0$, a compact extension $\tilde{T}$ from Z to X with $\|\tilde{T}\| \leqq (1+\epsilon)\|T\|$.*

(4) *Every operator T from Y to X whose range is of dimension $\leqq 3$ has, for every $\epsilon>0$, an extension $\tilde{T}$ from Z to X with $\|\tilde{T}\| \leqq (1+\epsilon)\|T\|$, provided* $\dim Z/Y=1$.

(5) *Every bounded operator T from Y to X has an extension $\tilde{T}$ from Z to X^{**} such that $\|\tilde{T}\|=\|T\|$. (X is embedded canonically in X^{**}.)*

(6) *Every bounded operator T from Y to X has an extension $\tilde{T}$ from Z to X such that $\|\tilde{T}\|=\|T\|$, provided that* $\dim Z/Y=1$ *and that $S_Z(0, 1)$ is the convex hull of $S_Y(0, 1)$ and a finite set of points.*

(7) *Every bounded operator T from X to a conjugate space Y has an extension $\tilde{T}$ from V to Y with $\|\tilde{T}\|=\|T\|$.*

(8) *X has the M.A.P. and every compact T from X to Y has a compact extension $\tilde{T}$ from V to Y with $\|\tilde{T}\|=\|T\|$.*

(9) *X has the M.A.P. and every weakly compact T from X to Y has a weakly compact extension $\tilde{T}$ from V to Y with $\|\tilde{T}\|=\|T\|$.*

(10) *X has the M.A.P. and every compact T from X into itself has an extension $\tilde{T}$ from V to X with $\|\tilde{T}\|=\|T\|$, provided that* $\dim V/X=1$.

(11) *For every finite collection of cells in X, such that any two of them intersect, there exists a point common to all the cells.*

(12) *For every collection of 4 cells in X, with radii equal to 1 and such that any two of them intersect, there is a point common to all the cells.*

For spaces X in which the unit cell has at least one extreme point the following statement is also equivalent to the preceding ones.

(13) *X is isometric to a subspace X_1 of some $C(K)$ (K compact Hausdorff) having the following properties:*

(a). *The function identically equal to 1 belongs to X_1.*

(b). *If $f, g, h \in X_1$ with $f, g, h \geqq 0$ and $f+g \geqq h$, then there are $f', g' \in X_1$ such that $0 \leqq f' \leqq f$, $0 \leqq g' \leqq g$ and $h=f'+g'$.*

For finite dimensional spaces X the equivalence of the properties (1)–(11) (except (4) and (6)) reduces to the finite dimensional case of the representation theorem of Nachbin, Goodner and Kelley for $\mathcal{P}_1$ spaces (see Day [**2**, p. 95] and Nachbin [**9**]). The equivalence of (11) and (12) for finite dimensional spaces was proved by Hanner [**5**]. The fact that (11)$\leftrightarrow$(12) also for infinite-dimensional spaces solves a problem raised by Aronszajn and Panitchpakdi [**1**]. It seems likely that it is possible to replace 3 by 2 in statement (4) but we did not succeed in proving this.

The spaces X satisfying (1)–(13) do not have in general the extension property (3) with $\epsilon=0$ (even if $\dim Y=2$, $\dim Z=3$). The question when it is possible to take $\epsilon=0$ is treated in

THEOREM 2. *Let X be a Banach space. The following statements are equivalent*:

(1) *Every operator T from Y to X whose range is of dimension* $\leqq 3$ *has a compact extension $\tilde{T}$ from Z, $Z\supset Y$, to X with* $\|\tilde{T}\|=\|T\|$.

(2) *Every operator T from Y to X with a finite-dimensional range has an extension $\tilde{T}$, with a finite-dimensional range, from Z, $Z\supset Y$, to X with* $\|\tilde{T}\|=\|T\|$.

(3) *X satisfies* (1)–(12) *of Theorem* 1 *and the unit cell of every finite-dimensional subspace of X is a polyhedron.*

The proof of this theorem is based on the ergodic theorem, Theorem 1 and a result of Klee [6]. As a by-product we obtained the following characterization of finite-dimensional spaces whose unit cell is a polyhedron:

Let B be a finite-dimensional space. The unit cell of B is a polyhedron if and only if for every $Z\supset B$ there is a function ϕ from B^* to Z^*, continuous in the norm topologies, such that the restriction of the functional $\phi(b^*)$ to B is equal to b^* and $\|\phi(b^*)\|=\|b^*\|$ for every $b^*\in B^*$.

3. **Examples and applications.** The spaces $C(K)$ have the properties (1)–(13) of Theorem 1. In fact these properties "almost" characterize $C(K)$ spaces. We have

THEOREM 3. *A Banach space X is isometric to a $C(K)$ space (K compact Hausdorff) if and only if it has the following three properties*:

(1) *X satisfies* (1)–(12) *of Theorem* 1.

(2) *The unit cell of X has at least one extreme point.*

(3) *The set of extreme points of the unit cell of X^* is w^* closed.*

No one of these three properties is implied by the other two. Clearly (1) is not implied by (2) and (3). The subspace of $C(0, 1)$ consisting of all the functions satisfying $f(0)+f(1)=0$ has the properties (1) and (3) but not (2). The space of the sequences $x=(x_1, x_2, \cdots)$ with $\lim x_i=(x_1+x_2)/2$ and $\|x\|=\max |x_i|$ satisfies (1) and (2) but not (3). This solves a problem raised by Nachbin [**8**; **9**].

No infinite dimensional $C(K)$ space has the properties appearing in Theorem 2. A simple example of a space having these properties is c_0.

Grothendieck [**4**] conjectured that a Banach space X has property (1) of Theorem 1 if and only if it is isometric to a subspace of some $C(K)$ consisting of all the functions f satisfying a set A of equations of the form $\lambda_\alpha f(k_\alpha^1)=\mu_\alpha f(k_\alpha^2)$ ($\alpha\in A$; λ_α, μ_α scalars; $k_\alpha^1, k_\alpha^2\in K$). We shall call such spaces G spaces (all M spaces [**2**, p. 100] and $C_\sigma(K)$ spaces [**2**, p. 89] are G spaces). It can be shown that every G space has the properties (1)–(12) appearing in Theorem 1 (it is easily verified that it satisfies (12)). On the other hand not every space satisfying (1)–(12) is a G space. Indeed, it can be proved that every G space whose unit cell has at least one extreme point is isometric to a $C(K)$ space. Hence the sequence space defined above is not a G space and this disproves the conjecture of Grothendieck.

We conclude the note with the following theorem (whose proof is based on Theorem 1) which solves a problem raised by Grünbaum and Semadeni.

THEOREM 4. *A space X which is a $\mathcal{P}_{1+\epsilon}$ space for every $\epsilon>0$, is also a $\mathcal{P}_1$ space.*

Detailed proofs of the theorems stated here and of various extensions of them are given in [7].

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