SOME L^p ESTIMATES FOR PARTIAL DIFFERENTIAL EQUATIONS[1]

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1. **Introduction.** In this note we describe sufficient conditions for inequalities of the form

$$\|u\|_{s,p} \leqq \text{const.}\left(\sum_k \|A_k u\|_{s_k,p} + \|u\|_{0,p}\right) \tag{1.1}$$

to hold for functions u satisfying given (possibly void) boundary conditions, where the A_k are linear partial differential operators, p is greater than one, and $\|\cdot\|_{s,p}$ is an L^p norm defined for all real values of s. When s is a non-negative integer, $\|u\|_{s,p}$ is essentially the sum of the L^p norms of u and all its derivatives up to order s.

We do not require that the A_k be of the same order or that s be greater than their maximum order. When s is an integer we also obtain inequalities of the form

$$\|u\|_{s,p} \leqq \text{const.}\left(\sum_k \|A_k u\|_{s_k,p} + \sum_j \langle B_j u\rangle_{t_j,p} + \|u\|_{0,p}\right) \tag{1.2}$$

holding for all functions u, where the B_j are boundary operators and the $\langle\cdot\rangle_{t,p}$ are appropriate boundary norms.

In the case of one operator A_1 we are able to obtain slightly stronger results (cf. §4).

We also consider corresponding inequalities for general bilinear integro-differential forms (cf. §5). If $[u, v]$ is such a form of order m, we give sufficient conditions for

$$\|u\|_{s,p} \leqq \text{const.}\left(\operatorname{lub}_v \frac{|[u,v]|}{\|v\|_{2m-s,p'}} + \sum_j \langle B_j u\rangle_{t_j,p} + \|u\|_{0,p}\right)$$

to hold for all u. This generalizes the concept of L^2 coerciveness for such forms.

2. **Complex interpolation spaces.** Let X_0 and X_1 be Banach spaces and denote by $H(X_0, X_1)$ the set of functions $f(x+iy)$ having values in X_0+X_1 which are analytic in $0<x<1$, continuous and bounded in $0\leqq x\leqq 1$, and such that $f(iy)\in X_0$, $f(1+iy)\in X_1$. Set

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$$\|f\|_{H(X_0,X_1)} = \max \left| \operatorname*{lub}_y \|f(iy)\|_{X_0}, \quad \operatorname*{lub}_y \|f(1+iy)\|_{X_1} \right| .$$

For $0 \leqq \theta \leqq 1$, the set $[X_0, X_1; \delta(\theta)]$ consists of those elements of $X_0 + X_1$ which are equal to $f(\theta)$ for some $f \in H(X_0, X_1)$. Under the norm

$$\|u\|_{[X_0,X_1;\delta(\theta)]} = \operatorname*{glb}_{f(\theta)=u} \|f\|_{H(X_0,X_1)},$$

the set $[X_0, X_1; \delta(\theta)]$ becomes a Banach space. This method of interpolation was introduced by Calderón [3] and Lions [4].

3. **Inequalities for formally positive forms.** Let G be a bounded domain in Euclidean n-space E^n with boundary ∂G of class C^∞. Let $C^\infty(\overline{G})$ denote the set of complex valued functions infinitely differentiable in the closure $\overline{G}$ of G. For i a non-negative integer and $p > 1$ we employ the norm

$$\|u\|_{i,p} = \left(\sum_{|\mu| \leq i} \int_G |D^\mu u|^p dx \right)^{1/p} \tag{3.1}$$

where summation is taken over all derivatives $D^\mu u$ of order $|\mu| \leqq i$. We let $H^{i,p}(G)$ denote the completion of $C^\infty(\overline{G})$ with respect to the norm (3.1). For any real number s such that $i < s < i+1$ we define $H^{s,p}(G)$ to be the space $[H^{i,p}(G), H^{i+1,p}(G); \delta(\theta)]$, where $\theta = s - i$. For s real and negative $H^{s,p}(G)$ is defined as the completion of $C^\infty(\overline{G})$ with respect to the norm,

$$\|u\|_{s,p} = \operatorname*{lub}_{v \in C^\infty(\overline{G})} \frac{|(u, v)|}{\|v\|_{-s,p'}},$$

where $(u, v) = \int_G u\bar{v}dx$ and $p' = p/(p-1)$.

We consider the following boundary norms. For $\phi \in C^\infty(\partial G)$ and s real and positive we define

$$\langle \phi \rangle_{s,p} = \operatorname{glb} \|u\|_{s+1/p,p},$$

where the glb is taken over all $u \in C^\infty(\overline{G})$ which equal ϕ on ∂G. For s negative, we write

$$\langle \phi \rangle_{s,p} = \operatorname{lub} \left| \int_{\partial G} \phi \bar{\psi} d\sigma \right| \langle \psi \rangle_{-s,p'}^{-1}, \tag{3.2}$$

where the lub is taken over all $\psi \in C^\infty(\partial G)$.

Let $\{A_k\}$ and $\{B_j\}$ be two finite systems of linear partial differential operators with coefficients in $C^\infty(\overline{G})$. The set $\{B_j\}$ may be void. Let m_k denote the order of A_k and ν_j the order of B_j. Set $m = \max m_k$

and $\nu = \max \nu_j$. We make the following assumptions:

(a) The orders of the B_j are distinct, and $\nu < m$.

(b) The boundary ∂G is noncharacteristic to each B_j at every point.

(c) At each point $x \in \overline{G}$ the characteristic polynomials $P_k(x, \xi)$ of the A_k do not vanish simultaneously for any real vector $\xi \neq 0$.

(d) The B_j cover the A_k. This means the following. At each point x^0 of ∂G let $N \neq 0$ be a vector orthogonal to ∂G at x^0 and $T \neq 0$ a tangential vector. Let $z_1, \cdots, z_h$ denote the complex roots with positive imaginary parts common to the polynomials $P_k(z) \equiv P_k(x^0, T + zN)$. If $Q_j(x, \xi)$ denotes the characteristic polynomial of B_j, then it is assumed that there are h polynomials among the $Q_j(z) \equiv Q_j(x^0, T + zN)$ which are linearly independent modulo the polynomial

$$(z - z_1)(z - z_2) \cdots (z - z_h).$$

If the set $\{B_j\}$ is empty, it is assumed that there are no such roots z_i.

(e) At each boundary point x^0, ∂G is noncharacteristic for some operator A_k of order m.

THEOREM 3.1. *Assume that the systems* $\{A_k\}$, $\{B_j\}$ *satisfy hypotheses* (a)–(e). *Then for each integer s and each set of real numbers $s_k \geqq s - m_k$, $t_j \geqq s - \nu_j - 1/p$ there is a constant C such that*

$$\|u\|_{s,p} \leqq C\left(\sum_k \|A_k u\|_{s_k,p} + \sum_j \langle B_j u \rangle_{t_j,p} + \|u\|_{s-m,p} \right) \tag{3.3}$$

for all $u \in C^\infty(\overline{G})$.

COROLLARY 3.1. *If $s \geqq m$, then hypothesis* (e) *is unnecessary in Theorem* 3.1.

THEOREM 3.2. *Under hypotheses* (a)–(e), *for every set of real $s \leqq m$ and $s_k \geqq s - m_k$ there is a constant C such that*

$$\|u\|_{s,p} \leqq C\left(\sum_k \|A_k u\|_{s_k,p} + \|u\|_{s-m,p} \right) \tag{3.4}$$

for all $u \in C^\infty(\overline{G})$ satisfying

$$B_j u = 0 \text{ on } \partial G \text{ for each } j. \tag{3.5}$$

Let m_0 be the minimum order of the A_k. Then an important special case of Theorem 3.2 is

COROLLARY 3.2. *Under hypotheses* (a)–(e),

$$\|u\|_{m_0,p} \leqq C\left(\sum_k \|A_k u\|_{0,p} + \|u\|_{0,p} \right)$$

for all $u \in C^\infty(\overline{G})$ satisfying (3.5).

REMARK 3.1. For s an integer and $\geqq m$, inequality (3.4) was first proved by Agmon (to appear). When the set $\{B_j\}$ is void, Smith [7] obtained the same result under stronger hypotheses on the A_k and weaker hypotheses on ∂G.

4. **A stronger result for one operator.** Assume that A is an elliptic operator of even order $2q$ with coefficients in $C^\infty(\overline{G})$. Let $\{B_j\}_{j=1}^q$ be a set of boundary operators satisfying hypotheses (a) and (b) of §3 (with $m=2q$) and having coefficients in $C^\infty(\partial G)$. If A' is the formal adjoint of A, let V' be the set of those $v\in C^\infty(\overline{G})$ such that $(u, A'v) = (Au, v)$ for all $u\in C^\infty(\overline{G})$ satisfying

$$B_j u = 0 \quad \text{on} \quad \partial G, \qquad 1 \leqq j \leqq q. \tag{4.1}$$

For a real s we define the norm

$$|w|'_{s,p} = \operatorname*{lub}_{v\in V'} \frac{|(w, v)|}{\|v\|_{-s,p'}}.$$

When $s\geqq 0$ this is equivalent to the norm $\|w\|_{s,p}$, but not otherwise.

THEOREM 4.1. *If the set $\{B_j\}_{j=1}^q$ covers*[2] *A, then for every real s there is a constant M_s such that*

$$\|u\|_{s,p} \leqq M_s(|Au|'_{s-2q,p} + \|u\|_{s-2q,p}) \tag{4.2}$$

for all u satisfying (4.1).

THEOREM 4.2. *Under the same hypothesis, for each integer s there is a constant M'_s such that*

$$\|u\|_{s,p} \leqq M'_s\left(|Au|'_{s-2q,p} + \sum_{j=1}^{q} \langle B_j u\rangle_{s-\nu_j-1/p,p} + \|u\|_{s-2q,p}\right) \tag{4.3}$$

for all $u\in C^\infty(\overline{G})$. Moreover, the same is true for all real $s\geqq \nu+1$.

REMARK. For the Dirichlet problem special cases of Theorem 4.1 and 4.2 were proved by Agmon [2] and Lions-Magenes [5]. For $p=2$ and general B_j the last statement of Theorem 4.2 is included in the work of Peetre [6].

5. **Bilinear forms.** We consider bilinear integro-differential forms of order m:

$$[u, v] = \int_G \sum_{|\mu|,|\tau|\leqq m} a_{\mu\tau} D^\mu u [D^\tau v]^- dx, \tag{5.1}$$

[2] Cf. hypothesis (d) of §3. In the present case there is only one operator in the set $\{A_k\}$.

where the coefficients $a_{\mu\tau}$ are in $C^\infty(\bar{G})$. Let $\langle B_j\rangle_{j=1}^r$ be a set of boundary operators satisfying hypotheses (a) and (b) of §3, and let V denote the set of those $u\in C^\infty(\bar{G})$ satisfying

$$B_j u = 0 \quad \text{on} \quad \partial G, \qquad 1 \leqq j \leqq r. \tag{5.1}$$

We say that the form $[u, v]$ is *coercive* over V if

$$\|u\|_{m,2}^2 \leqq \text{const.}(\text{Re}[u, u] + \|u\|_{0,2}^2)$$

for all $u\in V$. We have

THEOREM 5.1. *If* $[u, v]$ *is coercive over* V, *then for each real* $s\leqq m$

$$\|u\|_{s,p} \leqq \text{const.}([u]_{s-m,p} + \|u\|_{s-m,p}) \tag{5.2}$$

for all $u\in V$, *where*

$$[u]_{s-m,p} = \operatorname*{lub}_{v\in V} \frac{|[u, v]|}{\|v\|_{2m-s,p'}}.$$

Moreover, for each integer s

$$\|u\|_{s,p} \leqq \text{const.}\left([u]_{s-m,p} + \sum_{j=1}^{r} \langle B_j\rangle_{s-\nu_j-1/p,p} + \|u\|_{s-m,p}\right) \tag{5.3}$$

for all $u\in C^\infty(\bar{G})$.

Agmon [1] has given necessary and sufficient conditions for $[u, v]$ to be coercive over V. They are similar in nature to hypotheses (c) and (d) of §3. Hence we have obtained sufficient conditions for inequalities (5.2) and (5.3) to hold.

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