RESEARCH PROBLEMS

A. D. Wallace: *Problems concerning semigroups.*

A semigroup is a nonvoid Hausdorff space together with a continuous associative multiplication, generally denoted by juxtaposition. (General reference [2], in the discrete case [1].)

Very little is yet known concerning semigroups of matrices and in particular the following seems to be unsolved:

34. *Problem* 1. Is a compact connected semigroup of real $n \times n$ matrices with non-negative entries, which contains the unit matrix, necessarily acyclic? That is to say, do all of the cohomology groups (in positive dimensions) vanish?

A *clan* is a compact connected semigroup with unit. Generally, S will always denote a semigroup.

35. *Problem* 2. If S is a nondegenerate finite dimensional clan such that no retract of S cuts S, is S topologically an n-sphere, $n=1$ or 3?

A similar problem can be posed using the condition that S does not have the fixed point property but that every proper retract of S does have the fixed point property.

An element x of S is *periodic* if for some positive integer n we have $x^{n+1}=x$ and the least such n is the period, $p(x)$, of x. Moreover, S is *pointwise periodic* if each element of S has a period. We denote by E the set of idempotents of S.

36. *Problem* 3. If S is topologically an n-cell and if S is pointwise periodic, can it be so that $S \backslash E$ is nonvoid and that $p(x) > 2$ for each $x \in S \backslash E$ and that p is constant on $S \backslash E$?

It is easily shown that a pointwise periodic commutative clan is acyclic.

37. *Problem* 4. Is a pointwise periodic clan acyclic?

There seems not to exist even the beginnings of a representation theory for semigroups. The difficulty here is that compact semigroups rarely have invariant regular measures and hence that the function spaces on such semigroups are not Hilbert spaces.

38. *Problem* 5. If S is a compact semigroup, under what conditions on S will there exist a regular measure m on the Borel sets of S and a real function g on S such that

$$m(xA) = g(x)m(A)$$

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[1] Problems 34–38 were received by the editors May 2, 1962.

for each $x \in S$ and each Borel set A?

In studying the acyclicity properties of compact connected semigroups [3; 4] the following conditions have played a vital part:

(A) If A and B are closed subsets of E, then there is a subset C of E such that $AS \cap BS = CS$.

(B) If $a \in S$, then there is an element e in E such that $eSa = Sa$.

The first condition is satisfied if S is regular ($x \in xSx$ for each $x \in S$, see [1]). But both of the conditions are of pragmatic and *ad hoc* character and it would be of much interest to know the proper roles in the structure theory of semigroups.

In a somewhat vague way, it seems that semigroups appear more naturally in a physical universe than in a geometric universe. This, it may perhaps be supposed, is because groups are exemplars of reversible reactions while semigroups might be regarded as exemplars of irreversible reactions. The whole situation here being so obscure, a formal problem is not in order but it is worth framing the question as to what physical models there are for what clans.

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Mr. Dennison Brown kindly informs me that, working under auspices of Professor R. J. Koch, he has solved Problem 1 in the affirmative. Dr. A. L. Hudson informs me that she has solved Problem 4, also in the affirmative.

Bibliography

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