THE OPTIMAL LEBESGUE-RADON-NIKODYM INEQUALITY[1]

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The aim of the present paper is to sketch some further developments of order-integration (cf. [1; 2; 3]), and in particular to point out how the absence of lattice distributivity introduces some new and interesting aspects of the Lebesgue-Radon-Nikodym Theorem. Details will be published elsewhere.

The formula $\mu(x, y) = v(y) - v(x)$ establishes a 1-1 correspondence between the set of valuations of a lattice L (with identification modulo $v_1 \sim v_2 \Leftrightarrow v_1 - v_2 = \text{const}$) and the set L' of projectivity invariant, additive interval-functions ("quotient-functions") on L. If L is modular, then the equivalence classes of finite chains between x and y $(x \leqq y)$ form a directed set in virtue of the Schreier-Ore Theorem. Hence we may define Riemann-Darboux integrals of projectivity invariant interval-functions in the natural way. The R. D. integral of μ is additive (whenever it exists) and will be denoted by S_μ. Now the maximal directed vector subspace $L^* = (L')^+ - (L')^+$ of L' will consist of those $\mu \in L'$ which are of bounded variation in the sense that

$$(1) \qquad S_{|\mu|}(x, y) = \sup_{x = x_1 \leqq \cdots \leqq x_n = y} \sum_{i=1}^{n} | \mu(x_{i-1}, x_i) | < \infty;$$

for every interval (x, y). Moreover, L^* will be a conditionally complete vector lattice under the operations $\mu \vee \nu = S_{\mu \vee \nu}$, $\mu \wedge \nu = S_{\mu \wedge \nu}$. Decomposition of $\mu \in L^*$ in positive and negative parts yields the Jordan decomposition of μ (obtained by G. Birkhoff [4]).

The classical (Lebesgue-Vitali) definition of absolute continuity, $\nu \ll \mu$, for functions μ, ν on R can be directly transferred to the case in which μ, ν belongs to the space L^* of some modular lattice L. (The standard definition of $\nu \ll \mu$ for finitely additive measures μ, ν on a Boolean ring is obtained from the general definition by reduction of the chain involved to a two-interval chain by application of the Boolean difference available in this particular case.) The concept of mutual singularity, $\mu \perp \nu$, can be defined for members of L^* in an equally natural way. Let $\mathfrak{A}(\mu)$ denote the closed ideal ("famille com-

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plète," "bande") generated by $\mu \in L^*$. Then it can be proved that:

$$\nu \ll \mu \Leftrightarrow \nu \in \mathfrak{A}(\mu); \qquad \nu \perp \mu \Leftrightarrow \| \nu \| \wedge \| \mu \| = 0. \tag{2}$$

Hence F. Riesz' fundamental theorem on the decomposition of conditionally complete vector lattices into closed ideals [8; 6] yields a Lebesgue-decomposition in L^* (proved by H. Bauer [5]).

Let μ be an arbitrary, but fixed positive member of the space L^* of some modular lattice L. Then μ is a (weak) order unit of $\mathfrak{A}(\mu)$. Assume henceforth that L has a least element ϕ and a greatest element e. Then $\mathfrak{A}(\mu)$ is an abstract L-space in the sense of Kakutani [7] under the norm $N(\nu) = \|\nu\|(\phi, e) = S_{|\nu|}(\phi, e)$. Hence $\mathfrak{A}(\mu)$ is isomorphic to $L^1(S, \mathfrak{F}, m)$ where $(S, \mathfrak{F}, m)$ is an essentially unique, totally finite measure space whose measure algebra $\tilde{\mathfrak{F}}$ (i.e. $\mathfrak{F}$ modulo null sets) is canonically isomorphic to the complete Boolean algebra $\mathfrak{B}$ of closed subideals of $\mathfrak{A}(\mu)$. We call $(S, \mathfrak{F}, m)$ the *representation space* of $\mathfrak{A}(\mu)$, and we call the representative function $f_\nu \in L^1(S, \mathfrak{F}, m)$ of $\nu \in \mathfrak{A}(\mu)$ the *density function* of ν relative to μ.

For every $x \in L$ the annihilators

$$\mathfrak{H}_x = \{\nu \mid \nu \in \mathfrak{A}(\mu), \nu \equiv 0 \text{ on } [x, e]\},$$

$$\mathfrak{H}_x' = \{\nu \mid \nu \in \mathfrak{A}(\mu), \nu \equiv 0 \text{ on } [\phi, x]\}$$

are closed subideals of $\mathfrak{A}(\mu)$, $\mathfrak{H}_x \cap \mathfrak{H}_x' = (0)$, and $x \to \mathfrak{H}_x$, $x \to \mathfrak{K}_x = L^* \ominus \mathfrak{H}_x'$ are "meet"-preserving and "join"-preserving mappings of L into $\mathfrak{B}$, respectively. Let π^* be the "lifting" of the mapping $y \to x \wedge y$ from L to the set of (not necessarily projectivity invariant) interval functions. For every $\nu \in (L^*)^+$ the components $P_x\nu$, $Q_x\nu$ of ν into $\mathfrak{H}_x$, $\mathfrak{K}_x$ are the greatest and smallest projectivity invariant interval-functions, respectively, such that:

$$P_x\nu \leqq \pi^*\nu \leqq Q_x\nu. \tag{3}$$

From this one can conclude:

(i) $P_x\nu$ is the greatest member of L^* for which $0 \leqq P_x\nu \leqq \nu$ and $P_x\nu \equiv 0$ on $[x, e]$. On the other hand, $Q_x\nu$ is the smallest member of L^* for which $0 \leqq Q_x\nu \leqq \nu$ and $Q_x\nu \equiv \nu$ on $[\phi, x]$.

(ii) If x belongs to the distributive center Z of L (i.e. if x forms distributive triples with any two elements of L), then $y \to x \wedge y$ is lattice preserving, hence $\pi^*\nu$ is projectivity invariant, and so we have equality signs in (3). Thus $x \in Z$ implies $\mathfrak{H}_x = \mathfrak{K}_x$.

We denote the members of $\tilde{\mathfrak{F}}$ corresponding canonically to $\mathfrak{H}_x$, $\mathfrak{K}_x$, by H_x, K_x, and recall that the mappings $x \to H_x$, $x \to K_x$ will be "meet"-preserving and "join"-preserving, respectively, and that $H_x \subset K_x$, with $H_x = K_x$ whenever $x \in Z$. Now, we consider an arbitrary $\nu \in \mathfrak{A}(\mu)^+$ corresponding to a normalized valuation v (i.e. $v(\phi)=0$), and by the correspondence $v \leftrightarrow \nu$ we may write f_v instead of f_ν. By means of the explicit construction of the Kakutani representation and the possibility of a spectral resolution in $\mathfrak{A}(\mu)$, we can prove the following general *Lebesgue-Radon-Nikodym inequality*:

$$\int_{H_x} f_v dm \leqq v(x) \leqq \int_{K_x} f_v dm. \tag{4}$$

This result is in fact *optimal* in the sense that if

$$\int_A f_v dm \leqq v(x) \leqq \int_B f_v dm_H \tag{5}$$

for all $\nu \in \mathfrak{A}(\mu)$, then $A \subset H_x \subset K_x \subset B$. (Passage from the actual members of $\tilde{\mathfrak{F}}$ to representatives belonging to $\mathfrak{F}$, will of course transfer the last relations to inclusions modulo null-sets.) To verify this assertion, we consider the component ν of μ into the closed subideal $\mathfrak{A}$ corresponding (canonically) to A. Then $f_\nu = \chi_A$, and so $\nu(\phi, e) = \int f_\nu dm = m(A)$. By (5) we also have $\nu(\phi, x) \geqq \int_A f_\nu dm = m(A)$, and hence $\nu(x, e) = \nu(\phi, e) - \nu(\phi, x) \leqq 0$. By the positivity of ν this entails $\nu(x, e) = 0$, and in fact $\nu \equiv 0$ on $[x, e]$. In virtue of (i) we have $\nu \leqq P_x\mu$, and so $\mathfrak{A} \subset \mathfrak{H}_x$ proving that $A \subset H_x$. Similarly we can prove $K_x \subset B$.

It follows from (ii) that the distributive center Z of L is mapped homomorphically into $\tilde{\mathfrak{F}}$ by $x \rightarrow H_x(=K_x)$, and that:

$$v(x) = \int_{H_x} f_v dm \tag{6}$$

for $\nu \in \mathfrak{A}(\mu)$ and $x \in Z$. (Actually (6) remains valid even if we only require the equivalence class $\bar{x}$ of x modulo v to belong to the distributive center of $Z/[v]$.) In particular, *we shall have an exact Lebesgue-Radon-Nikodym Theorem* (6) *on distributive lattices.*

If L is taken to be a Boolean algebra, then v becomes a finitely additive measure (recall $v(\phi)=0$), and the mapping $x \rightarrow H_x$ will be a representation (but not necessarily a σ-representation) of L into $\mathfrak{B}$. If L is taken to be the Boolean algebra $\mathcal{G}$ of some totally finite measure space $(T, \mathcal{G}, p)$ and $v = p$, then the σ-continuity of v will imply that $x \rightarrow \mathfrak{H}_x(=\mathfrak{K}_x)$ is a σ-representation, and hence so is $x \rightarrow H_x(=K_x)$. From this one can conclude that $(T, \mathcal{G}, p)$ is one of the possible realizations of the (essentially unique) representation space. Thus we obtain the standard Lebesgue-Radon-Nikodym Theorem in this case.

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