ON THE TANGENTIAL PROPERTIES OF SURFACES

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Communicated by E. J. McShane, January 9, 1962

It is the task of this paper to generalise certain results of [1] concerning the structure of surfaces. The motivation is in part to extend the solution [2] of the Plateau problem from the case of Hausdorff spherical measure to other forms of "area," particularly Hausdorff convex measure.

We consider the class of surfaces defined in [2]. $L \subset H_{m-1}(A)$ and S is a surface with boundary $\supset L$. We write $\mu(L)$ for the minimum area $\Lambda^m(S-A)$ of surfaces with boundary $\supset L$. We also write $\mu(A)$ for $\mu(H_{m-1}(A))$.

Let $S(P, r)$ denote the solid sphere of centre P and radius r, and denote its surface by $s(P, r)$. Let T be the set of points of $S^0 = S - A$ where S is (Λ^m, m)-restricted [3] and

$$\frac{\mu(s(P, r)S)}{r^m} \not\to 0.$$

At these points we say there is an approximate tangential plane. Let

$$\mathfrak{A}(S) = \lim_{\epsilon \to 0} \inf_{\Omega_\epsilon} \sum \mu(S(\overline{G}_i - G_i))$$

where Ω_ϵ is the class of sets of nonoverlapping open sets G_i of diameter less than ϵ not meeting A and such that $\Lambda^m(S^0 - \sum G_i) = 0$.

We shall show that:

THEOREM.

$$\Lambda^m T = \mathfrak{A}(S).$$

As regards applying this to the Plateau problem for Hausdorff convex measure, ${}_c\Lambda^m$, it suffices to note that for a surface minimising Hausdorff spherical measure, Λ^m, the two measures are equal and in any case

$$\Lambda^m(S^0) \geqq {}_c\Lambda^m(S^0) \geqq {}_c\Lambda^m(T) = \Lambda^m(T) = \mathfrak{A}(S) \geqq \mu(L).$$

It is perhaps worth noting that $\mathfrak{A}(S)$ need not be as great as $\mu(A)$.

In order to prove that $\Lambda^m(T) = \mathfrak{A}(S)$ it suffices to cover nearly all of $S^0 - T$ with small nonoverlapping open G_i such that $\sum \mu(S(\overline{G}_i - G_i))$ is small. We do so by applying the Vitale general covering principle [4] in two stages. At the first stage consider those points of S^0 which are restricted but not in T. By definition

$$\frac{\mu(s(P, r)S)}{r^m} \to 0 \quad \text{but} \quad \frac{\Lambda^m SS(P, r)}{r^m} \nrightarrow 0$$

so that there exist arbitrarily small spheres on which

$$\frac{\mu\{s(P, r)S\}}{\Lambda^m SS(P, r)} < \epsilon.$$

At the second stage we must consider the unrestricted points. To do so we need the full power of the main result of [3]; the "projection theorem."

According to this any m-set E can be divided into two parts the restricted part E_R and the unrestricted part E_u and whereas the projection of E_R onto almost any m plane is a set of positive measure the projection of E_u onto almost any m plane is a set of measure zero.

In n-dimensional Euclidean space let $E(\theta, h)$ be the part of E on the $(n-1)$-plane at distance h from the origin and normal to the direction θ. Write $R_m E = \Lambda^m E_R$. Then:

LEMMA 1. *For almost all* θ

$$R_m(E) \geqq \int R_{m-1}(E(\theta, h))dh.$$

We can split E up into its restricted and unrestricted parts. For a restricted set [2]

$$R_m(E) = \Lambda^m(E) \geqq \int \Lambda^{m-1} E(\theta, h)dh \geqq \int R_{m-1} E(\theta, h)dh,$$

and so we can confine ourself to the case of an unrestricted set E when we must prove that

$$\int R_{m-1}(E(\theta, h))dh = 0. \tag{1}$$

Let Π_m be an m-plane through the origin, θ a vector in Π_m and h a real variable on θ. Write $f(\theta, \Pi_m, h)$ for the $(m-1)$-dimensional Hausdorff spherical measure of the projection of $E(\theta, h)$ onto Π_m.

$$\int f(\theta, \Pi_m, h)dh = \Lambda^m \text{ (Projection onto } \Pi_m \text{ of } E)$$

and therefore for almost all Π_m the left-hand side is zero whenever $\theta \subset \Pi_m$.

Thus taking a fixed plane Π^0_{n-m+1}

$$\int_{\theta\subset\Pi^0_{n-m+1}} d\theta \int_{\Pi_m\supset\theta} d\Pi_m \int f(\theta, \Pi_m, h)dh = 0.$$

Hence for almost all $\theta \subset \Pi^0_{n-m+1}$

$$\int_{\Pi_m\supset\theta} d\Pi_m \int f(\theta, \Pi_m, h)dh = 0$$

i.e.

$$\int dh \int_{\Pi_m\supset\theta} d\Pi_m f(\theta, \Pi_m, h) = 0$$

so that for almost all h: $f(\theta, \Pi_m, h) = 0$ for almost all $\Pi_m \supset \theta$, which implies $R_{m-1}E(\theta, h) = 0$. This is true for almost all $\theta \subset \Pi^0_{n-m+1}$ and hence for almost all θ.

Thus for almost all θ (1) holds and the lemma follows.

LEMMA 2. *If Γ is an $(m-1)$ dimensional boundary of diameter d and finite measure then*

$$\mu(\Gamma) \leqq \frac{d\mathfrak{A}(\Gamma)}{m}\cdot$$

Choose a nonoverlapping set of open G_i of diameter less than $\epsilon > 0$ such that

$$\sum_1^\infty \mu((\overline{G}_i - G_i)\Gamma) \leqq \mathfrak{A}(\Gamma) + \epsilon \tag{2}$$

and

$$\Lambda^{m-1}\left(\Gamma - \sum_1^\infty G_i\right) = 0. \tag{3}$$

Let S_i be a surface of diameter less than ϵ with boundary $\supset H_{m-2}((\overline{G}_i - G_i)\Gamma)$ and of measure $\mu((\overline{G}_i - G_i)\Gamma)$. Then for any N by [2]

$$\mu(\Gamma) \leqq \mu\left(\Gamma - \sum_1^N G_i + \sum_1^N S_i\right) + \sum_1^N \mu(S_i + \overline{G}_i\Gamma)$$

whence for large N (3) and Lemma 7 of [2] give

$$\mu(\Gamma) \leqq \frac{d(\mathfrak{A}(\Gamma) + 2\epsilon)}{m} + \frac{\epsilon\cdot 2\Lambda^{m-1}(\Gamma)}{m}$$

from which the lemma follows by letting $\epsilon \to 0$.

I now proceed by induction. Any closed set of finite linear measure can [5] be divided into a totally disconnected set plus a denumerable set of arcwise connected sets; so that our main theorem holds for $m=1$.

Suppose then that for $(m-1)$-dimensional surfaces $\Lambda^{m-1}(T)=\mathfrak{A}(S)$ so that by Lemma 2

$$\mu(\Gamma) \leqq \frac{dR_{m-1}(\Gamma)}{m}. \tag{4}$$

Take G an open set containing S_u^0 the unrestricted part of S^0 and such that

$$R_m(GS^0) \leqq R_m(S_u^0) + \epsilon = \epsilon. \tag{5}$$

If $P \in S_u^0$ then by virtue of Lemma 1 we can find arbitrarily small cubes $\Delta(h) \subset G$ with side $2h$ and centre P such that if $\delta(h)$ denotes the surface of $\Delta(h)$ then

$$\frac{2R_m(S\Delta(h_1))}{h_1} \geqq R_{m-1}(S\delta(h_2))$$

and $h_1 > h_2 > h_1/2$.

By (4) above

$$\mu(S\delta(h_2)) \leqq \frac{2n}{m} R_m(S\Delta(h_1)). \tag{6}$$

By Vitale's theorem [4] we can now find a covering almost all of S_u^0 by a set ϕ of nonoverlapping $\Delta(h_2)$. The corresponding $\Delta(h_1)$ can be divided into a bounded set of classes of nonoverlapping cubes (cf. [4]) and each $\Delta(h_1) \subset G$; consequently by (5) and (6)

$$\sum_{\phi} \mu(S\delta(h_2))$$

is small and the theorem follows.

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