ON GROUPS WITH FINITELY MANY INDECOMPOSABLE INTEGRAL REPRESENTATIONS

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1. **Introduction.** The purpose of this note is to sketch a proof of the following theorem.

THEOREM. *If G is a finite group having finitely many non-isomorphic indecomposable integral representations then for no prime p does p^3 divide the order of G.*

It is known that the same hypothesis implies that all the Sylow subgroups of G are cyclic; thus they are cyclic of order p or p^2. We do not know whether the converse is true. On the other hand, we have shown elsewhere [1] that a cyclic group of order p^2 has finitely many non-isomorphic integral representations.

In the same place it is shown that the above theorem follows from this proposition:

PROPOSITION. *Let G be a cyclic group of order p^3. Then G has infinitely many non-isomorphic indecomposable representations over the p-adic integers.*

We outline below the proof of this proposition, which will appear in full elsewhere.

2. **Construction of indecomposables.** Let Λ be a ring such that the Krull-Schmidt theorem holds for finitely generated left Λ-modules; this is certainly the case for algebras of finite rank over a complete valuation ring [3]. We shall write Hom for Hom_Λ and Ext for Ext^1_Λ.

Suppose that M and N are indecomposable Λ-modules such that $\mathrm{Hom}(M, N)=0$, $\mathrm{Hom}(N, M)=0$. If $M^{(k)}$ is a direct sum of k copies of M then $\mathrm{Hom}(M^{(k)}, M^{(k)})$ may be identified with the ring of $k\times k$ matrices with entries in $H=\mathrm{Hom}(M, M)$. Also $\mathrm{Ext}(N^{(u)}, M^{(t)})$ consists of $t\times u$ matrices with entries in $\mathrm{Ext}(N, M)$. If $H'=\mathrm{Hom}(N, N)$ then $\mathrm{Ext}(N, M)$ is an (H, H')-bimodule, and $t\times t$ matrices over H and $u\times u$ matrices over H' operate in the obvious way on $\mathrm{Ext}(N^{(u)}, M^{(t)})$.

We shall say that a matrix $X\in\mathrm{Ext}(N^{(u)}, M^{(t)})$ is decomposable if there are invertible matrices T over H and U over H' such that

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$$T \times U = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix},$$

where, of course, B and D need not be square matrices.

LEMMA 1. *An extension E of $N^{(u)}$ by $M^{(t)}$ with extension class X is a decomposable module if and only if X is a decomposable matrix.*

In order to apply this lemma it is convenient to observe the following consequence.

COROLLARY. *Let $\tilde{H}$, $\tilde{H}'$ be quotient rings of H, H'. Suppose $V \subset \mathrm{Ext}(N, M)$ is an (H, H')-submodule and that $\tilde{V}$ is a quotient of V on which $\tilde{H}$, $\tilde{H}'$ operate. If X is a matrix with entries in V whose image $\tilde{X}$ in $\tilde{V}$ is $(\tilde{H}, \tilde{H}')$-indecomposable then the extension corresponding to X is an indecomposable module.*

3. **Construction of the submodule.** In this paragraph we set $\Lambda = E_2 = Z_p^* G_{p^2}$, where Z_p^* is the ring of p-adic integers, and G_{p^2} is cyclic of order p^2 with generator g. We write $C = (g^p - 1)E_2$ and $E_1 = E_2/C$. For any module N, we shall set $\overline{N} = N/pN$.

Now Ext $(C, E_1) \approx \overline{E}_1 \approx \overline{Z}[g]/(g-1)^p$. We define M to be the extension of C by E_1 with extension class $g-1$. Since $\mathrm{Hom}(E_1, C) = 0$, $\mathrm{Hom}(C, E_1) = 0$, we may apply Lemma 1 with $k = 1$. Thus M is indecomposable. Further, if $H = \mathrm{Hom}(M, M)$, there is a canonical monomorphism $\rho: H \to \mathrm{Hom}(C, C) \dotplus \mathrm{Hom}(E_1, E_1)$ whose image may be described as follows [2].

LEMMA 2. *$\rho(H)$ consists of pairs (a_L, b_L), where $a, b \in E_2$ and the subscript L denotes left multiplication, such that*

$$(g - 1)(a - b) \in pE_2 + (g - 1)^p E_2.$$

Denoting by rad H the Jacobson radical of H, we have the following consequence.

COROLLARY. *$\rho(\mathrm{rad}\ H)$ consists of pairs $(a_L, b_L) \in \rho(H)$ such that $a, b \in \mathrm{rad}\ E_2 = pE_2 + (g-1)E_2$. Thus $\tilde{H} = H/\mathrm{rad}\ H \approx \overline{Z}$.*

Although M is indecomposable this is not true of $\overline{M}$. We have instead the following result.

LEMMA 3. *$\overline{M} = E_2 u \oplus E_2 v$ as an E_2 module, where $pu = pv = (g-1)u = (g-1)^{p^2-1} v = 0$.*

Now let V be the submodule $E_2 u + E_2(g-1)v$ of $\overline{M}$. Then, as a consequence of Lemma 2, we have the following result.

LEMMA 4. *V is an H-submodule of $\overline{M}$ and* (rad H) $V = E_2(g-1)^2 v$. *Thus $\tilde{V} = V/$*(rad H) *V is a two-dimensional $\tilde{H}$-space with basis $\tilde{u}$, $\tilde{v}$, the images of u and* $(g-1)v$.

4. **Proof of the proposition.** We now change our notation so that $\Lambda = E_3 = Z_p^* G_{p^3}$ where G_{p^3} is cyclic of order p^3 with generator g_3. Then $g_3 \to g$ defines a ring epimorphism $E_3 \to E_2$; we use this to turn all E_2-modules into E_3-modules.

If $N = (g_3^{p^2} - 1)E_3$, and M is the module defined in §3, then $\mathrm{Hom}(M, N) = \mathrm{Hom}(N, M) = 0$ and $\mathrm{Ext}(N, M) \approx \overline{M}$. But $H' = \mathrm{Hom}(N, N)$ consists only of left multiplications a_L, $a \in E_3$. Thus (rad H') $V = E_2(g-1)^2 v$ and $\tilde{H}' = H'/\mathrm{rad}\, H' \approx \overline{Z}$ operates on $\tilde{V}$.

We are now in a position to apply the corollary to Lemma 1. For any integer k let $X^{(k)} \in \mathrm{Ext}(N^{(k)}, M^{(k)})$ be the matrix $X^{(k)} = uI + (g-1)vJ$, where J is any $k \times k$ indecomposable matrix over $\overline{Z}$. Since the matrices $\tilde{X}^{(k)} = \tilde{u}I + \tilde{v}J$ are clearly $\overline{Z}$-indecomposable, i.e., $(\tilde{H}, \tilde{H}')$-indecomposable, the same must be true of the corresponding extensions.

REFERENCES

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