ENTIRE FUNCTIONS AND INTEGRAL TRANSFORMS

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If $E(z)$ is an entire function which satisfies

$$|E(\bar{z})| < |E(z)| \tag{1}$$

for $y>0$ $(z=x+iy)$, let $\mathcal{H}(E)$ be the corresponding Hilbert space of entire functions $F(z)$ such that

$$\|F\|^2 = \int |F(t)/E(t)|^2 dt < \infty$$

and

$$|F(z)|^2 \leqq \|F\|^2[|E(z)|^2 - |E(\bar{z})|^2]/[2\pi i(\bar{z} - z)]$$

for all complex z. The space is introduced in [**7**], where it is characterized by three axioms. If $E(a, z)$ and $E(b, z)$ are entire functions which satisfy (1), then $\mathcal{H}(E(a))$ will be contained isometrically in $\mathcal{H}(E(b))$ if these functions satisfy the hypotheses of Theorem VII of [**8**]. Isometric inclusions of spaces of entire functions are a basic idea in [**9**] and [**10**]. A fundamental property of these inclusions has only now become available.

THEOREM I. *If $E(a, z)$, $E(b, z)$, and $E(c, z)$ are entire functions which satisfy* (1) *and have no real zeros, and if $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ are contained isometrically in $\mathcal{H}(E(c))$, then either $\mathcal{H}(E(a))$ contains $\mathcal{H}(E(b))$ or $\mathcal{H}(E(b))$ contains $\mathcal{H}(E(a))$.*

The formal proof depends on techniques of [**2**] and [**3**] for handling difference quotients. To make it precise, one must show that if $f(z)$ and $g(z)$ are entire functions of minimal exponential type such that

$$|yf(z)g(z)| \leqq |f(z)| + |g(z)|$$

for all complex z, then $f(z)g(z)$ vanishes identically. This is proved by a method of Carleman, for whose explanation we are indebted to M. Heins [**16**]. By Theorem III of [**10**], the theorem has applications for certain kinds of integral transforms.

THEOREM II. *Let $u(x)$ and $v(x)$ be square integrable functions defined in* $[0, 1]$, *such that*

$$\bar{u}(x)v(x) = \bar{v}(x)u(x)$$

a.e., and which are essentially linearly independent when restricted to *any subinterval of* [0, 1]. *Let* T *be the bounded linear transformation of* $L^2(0, 1)$ *into itself defined by* $T: g \to f$ *if*

$$f(x) = \int_x^1 g(t)[u(x)\bar{v}(t) - v(x)\bar{u}(t)]dt$$

for almost all values of x. *Let* $\mathfrak{M}$ *be a closed subspace of* $L^2(0, 1)$ *which is invariant under* T *in the sense that* Tg *belongs to* $\mathfrak{M}$ *whenever* g *belongs to* $\mathfrak{M}$. *Then,* $\mathfrak{M}$ *is characterized by a number* a *in* [0, 1] *and coincides with the set of functions which vanish a.e. for* $x \geqq a$.

The same conclusion is available from the work of Kalisch [17] when $u(x)$ and $v(x)$ satisfy additional differentiability conditions. The point of Theorem II is that no such restrictions are necessary. Theorem II may be used to give a proof of uniqueness in the inverse Sturm-Liouville problem studied by Levinson [19].

THEOREM III. *Let* $\psi(x)$ *be a uniformly continuous, increasing function of real* x *such that*

$$\int (1 + t^2)^{-1} |\psi(t) - \tau t|^2 dt < \infty$$

for some number $\tau > 0$. *If* $0 < a < \tau$, *then there is a measure* μ *of finite total variation, supported in the points* t *where* $\psi(t) \equiv 0$ *modulo* π, *such that* $\int e^{ixt} d\mu(t)$ *vanishes in* $[-a, a]$ *and does not vanish identically. Furthermore, the measure may be chosen of this special form: There is an entire function* $S(z)$ *of exponential type* a *which is real for real* z *and has only real simple zeros, all at points* t *where* $\psi(t) \equiv 0$ *modulo* π, *and*

$$\int (1 + t^2)^{-1} \log^+ |S(t)| \, dt < \infty \tag{2}$$

and

$$\sum_{S(t)=0} |S'(t)|^{-1} < \infty.$$

The measure μ *is supported in the zeros of* $S(z)$ *and has mass* $S'(t)^{-1}$ *at each such zero* t.

The formal part of the proof depends on the formula of [6] to obtain a measure, and on the convexity methods of [4] and [5] to obtain an entire function. To implement these procedures, we use a theorem of Beurling and Malliavin [20]: If $K(z)$ is an entire function of exponential type which satisfies (2), then for each $a > 0$ there is a nonzero entire function $F(z)$ of exponential type a, bounded on the real axis, such that $K(z)F(z)$ is bounded on the real axis. Under the hypotheses of Theorem III, an entire function of minimal exponential type, which remains bounded on the set of points t where $\psi(t) \equiv 0$ modulo π, is necessarily a constant. We should like to acknowledge our indebtedness to Chapter VIII of Levinson [18], which suggested the above theorem. The results of Levinson, Chapter IX, can be significantly bettered on using another theorem of Levinson, as it is formulated in [3]. The trick is to use Theorem XII of [9] to convert a result on nonvanishing Fourier transforms into an existence theorem for entire functions of minimal exponential type.

THEOREM IV. *Let* (a_n, b_n) *be a sequence of disjoint intervals to the right of* $x=1$ *with lengths* $b_n - a_n$ *bounded away from zero and with*

$$\sum (b_n - a_n)^2 a_n^{-1} b_n^{-1} = \infty.$$

Then there exists an entire function of minimal exponential type which remains bounded on the real complement of $\bigcup(a_n, b_n)$ *and is not a constant.*

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