BOOK REVIEWS

Transzendente Funktionen. By A. Kratzer and W. Franz. Akademische Verlagsgesellschaft, Leipzig, 1960. XIII+375 pp. DM 39.

The "transcendental functions" presented in this book are the hypergeometric and confluent hypergeometric functions and those special functions closely related to them. Thus, the material covered in the present book is roughly the same as that covered in the first half of "Part II. The Transcendental Functions" of Whittaker and Watson's classic, *Modern analysis*.

All the functions studied in the book are related to Gauss' hypergeometric series through special and limiting cases, and the presentation is further unified by basing all work, as far as possible, on contour integral representations of the functions concerned. There is much to be said in favour of such an organization, and it will certainly be welcomed by the student who wishes to work through the whole book. It must be admitted, though, that this plan of organization makes it virtually impossible to read the chapter on Bessel functions without having read the chapter on confluent hypergeometric functions, and in its turn the latter chapter is difficult to read without a knowledge of the basic chapter on the hypergeometric functions. Also, the emphasis on integral representations leads not only to the exclusion of functions which do not possess integral representations, but also to a concentration on integrals of the Euler-Laplace type. Contour integrals involving gamma functions (Mellin-Barnes integrals) do not find their natural place in the authors' scheme, and hypergeometric functions represented by such integrals (MacRobert's E-function, Meijer's G-function) are left on one side. Furthermore, those questions not particularly suited to the contour integral method are underplayed.

These features are mentioned here not so much in a spirit of criticism as in an attempt to define the scope of the book. It might be said that the book is a thorough study in the integration of the hypergeometric and confluent hypergeometric equations by contour integrals of Euler's and Laplace's type, and an investigation of the special functions arising in this work. The pace of the book is leisurely, motivation and proofs are given fully, and the authors make great, and successful, efforts to explain the tricky technique of contour integration on Riemann surfaces of integrands with branch points. Their aim appears to be to teach methods as much as to present results. A student having a firm grasp of advanced calculus and a good knowledge of the more elementary parts of complex variable analysis will, after having worked through this book of less than 400 pages, not only have acquired a knowledge of the most important properties of the "transcendental functions," but will have also acquired a thorough mastery of the techniques used for investigating these functions.

The first three chapters are of an introductory nature, comprising between one quarter and one fifth of the book.

Chapter 1 is devoted to the beta and gamma functions. This chapter contains a discussion of integrals of the form

$$\int_a^b (z - a)^{\alpha-1}(z - b)^{\beta-1}f(z)dz$$

where $f(z)$ is single-valued, integration is extended either over an arc joining a and b, a loop starting at one of these points and encircling the other, or a Pochhammer double loop slung around the two points a and b. The careful discussion of analytic continuation (in α and β) and of the exceptional cases of integer α, β or $\alpha \pm \beta$ is worthy of special commendation.

Chapter 2 contains the classification of singularities of linear ordinary differential equations with analytic coefficients, Fuchsian equations of order two and their transformations, the construction of the hypergeometric equation and its (linear) transformations, and a study of the confluence of the hypergeometric equation, in case two singularities coalesce.

Chapter 3 introduces confocal coordinates and their special and limiting cases (spherical polar, cylindrical, paraboloidal, spheroidal, sphero-conal, elliptic-cylindrical, and parabolic-cylindrical coordinates). The separation of the wave equation in these coordinates is carried through, and the ordinary differential equations arising in the separation are briefly discussed. The reviewer has one reservation about this otherwise very lucid chapter. The even tenor of presentation might mislead a beginner into believing that all the ordinary differential equations arising in this work are, roughly speaking, equally difficult and equally well known. Since no references or explanations are given, he may fail to realize that much is known about, say, Legendre's equation, while very little information is available on the differential equation of Lamé wave functions.

Chapter 4 might be regarded as the backbone of the book, devoted as it is to Gauss' hypergeometric function. The integration of the hypergeometric equation by Eulerian integrals is carried out in detail, leading to the introduction of the hypergeometric series, construction of fundamental systems of solutions, analytic continuation of the hypergeometric series, relations between contiguous functions, and the like. As in the case of the beta function, the careful discussion of several special cases, notably of integer values of the parameters or their combinations, is particularly noteworthy. The discussion of the asymptotic behaviour of the hypergeometric function for large values of the parameters, based on the method of steepest descents, is probably the most complete discussion of this kind to be found in any textbook. On the other hand, the asymptotic forms valid in the neighbourhood of the transition point (and of importance in some problems of applied mathematics) are not even mentioned. Jacobi and Legendre polynomials are briefly presented, and the chapter contains also a brief reference to Appell's and Lauricella's hypergeometric functions of two and several variables (but no reference to Horn's series in two variables). Generally speaking, the student working through this chapter learns practically nothing about the more "theoretical" aspects of the subject, such as the theory of quadratic and cubic transformations, the group of the equation, conformal mapping of curvilinear triangles, automorphic functions, continued fractions, and does not meet the Mellin-Barnes type integral representations; but he does acquire a considerable facility in the analytical manipulation of hypergeometric functions and has readily available all the material needed later for the study of Legendre and Bessel functions.

Legendre functions are presented in Chapter 5. They are defined in terms of hypergeometric functions, and their properties are derived from the results established in Chapter 4. As in other parts of the book, the discussion of branch points and the appropriate determination of the many-valued functions is carried through in much greater detail than in many other textbooks. Integral representations are also thoroughly discussed as are asymptotic expansions for large degree or large order. Spherical surface harmonics are briefly mentioned but, somewhat disappointingly, Maxwell's generation of these and the integral equation which they satisfy are not. On the other hand, the discussion of the addition theorems is unusually detailed and complete. The last section of this chapter is devoted to Laplace's equation in n-dimensional space and to the Gegenbauer functions arising in its separation in hyperspherical coordinates.

Chapter 6 is devoted to confluent hypergeometric functions. The differential equation of these obtained by letting two singularities of the hypergeometric equation coalesce. This equation is then integrated by Laplace integrals which are shown to be limiting cases of the Eulerian integrals used in the solution of the hypergeometric equation. Whittaker's symmetric form of the confluent hypergeometric equation is also obtained. Analytic continuation of the solution, convergent and asymptotic expansions, relations between contiguous functions, and the corresponding relations for Whittaker functions, are all obtained from the integral representations. Laguerre and Hermite polynomials, and other particular confluent hypergeometric functions (parabolic cylinder functions, incomplete gamma functions, error functions, Fresnel integrals, exponential integral, sine and cosine integrals) follow. There is also a section on the occurrence of confluent hypergeometric functions in connection with Laplace transforms.

The longest chapter of the whole book (a little over one quarter of the entire work) is Chapter 7, which is devoted to Bessel functions. Bessel's equation is transformed into the confluent hypergeometric form, and Bessel functions are defined in terms of confluent hypergeometric functions, whereupon a wealth of formulas at once becomes available. These include series expansions, recurrence relations, and a multitude of integral representations. Sommerfeld's integrals are discussed in great detail, corresponding to their great importance in wave propagation problems. Integrals involving Bessel functions follow and lead to a discussion of the connection between Bessel functions and Legendre functions. An especially thorough section is devoted to zeros of Bessel functions, in particular to results of a nature likely to be useful in applications to boundary value problems. The investigation, by the method of steepest descents, of the asymptotic behaviour of Bessel functions of large variable and large order is carried out for complex variable z and complex order p. The authors attempt to present the notoriously involved results in a perspicuous form by determining the curves, in the $\cos^{-1}(p/z)$ plane, along which two saddle points give contributions of equal modulus; these curves they regard as branch cuts for the asymptotic representations. They discuss also the formulas of Nicholson and Schöbe valid near the transition point, but they do not mention the uniform asymptotic expansions of Langer, Cherry, and Olver (perhaps because these are obtained from the differential equation rather than the integral representation of Bessel functions). Addition theorems and other series and integral expansions are treated rather more briefly. The chapter (and the book) closes with a brief account of some of the functions related to Bessel functions.

The bibliography appended to the book is somewhat disappointing. The more important compendia are all listed, but on individual functions the reader does not get full information. For instance, the only book in the list on Legendre functions is that by Hobson (1931); the more recent books by MacRobert (1947), Lense (1950), Robin (1957–1959) are not brought to the reader's attention.

The general impression one gets of this book is that of a thorough and very lucidly written textbook which is well suited to systematic study from cover to cover and will reward the student not only with a knowledge of the functions presented in the book but also with an unusually clear presentation of the various methods used in acquiring this knowledge. All in all a valuable addition to the growing number of books devoted to special functions.

A. ERDÉLYI

Introduction to geometry. By H. S. M. Coxeter. John Wiley & Sons, New York, London, 1961. 15+443 pp. $9.95.

Geometry, in the Greek view synonymous with mathematics, was confronted with two competitors when mathematics revived in modern times, with algebra and analysis. But though more powerful the methods of algebra and analysis were still considered as philosophically inferior to that of geometry. And mathematical rigor remained associated with the names of Euclid and Archimedes. It is a fact that today geometry has lost much of its reputation. Geometry pursued along traditional lines is called old-fashioned, not only because of an apparent lack of rigor, but also on account of the alleged insignificance of its results. Coxeter presents classical style geometry in 22 chapters, which are reasonably self-contained, though tied together by a modern spirit of reinterpretation of classical matter. If geometry can be rewritten in a modern style without losing its classical character, is it fair to call it out of date? The answer of dogmatics to this rhetorical question will still be: yes, it is. They will emphasize this answer when they read the table of contents of the first chapter "Triangles": 1. Euclid, 2. Primitive concepts and axioms, 3. *Pons asinorum*, 4. The medians and the centroid, 5. The incircle and the circumcircle, 6. The Euler line and the orthocenter, 7. The nine-point circle, 8. Two extremum problems, 9. Morley's theorem. Of course they will never read this chapter (or the others either). If they are endowed with a sense of mathematical beauty, this is to be regretted. Fortunately there are people left, who like mathematical still-life. If they read this chapter they will admire not only the choice of subjects, but also the condensed style as opposed to the verbosity of many older geometry texts, and the compact lucid proofs in which every definition and conclusion is completely to the point. These are