RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

COVERINGS IN FREE LATTICES

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The purpose of this note is to exhibit some new coverings in the free lattice, FL(n), generated by a finite number of elements. Free lattices were studied extensively by P. M. Whitman [1; 2] who found a number of coverings. Theorem 1 of this note guarantees an infinite number of distinct pairs of covering elements.

Let FL(n) have generators $x_1, \cdots, x_n$. The definition of the words (elements) of the lattice and the ordering of the words are those of Whitman [1]. We make use of his Lemma 1.1 appearing in [2]:

In FL(n), if w is any word and x_r is any generator, then either $w \geqq x_r$ or $\bigcup_{i\neq r} x_i \geqq w$, but not both.

LEMMA. *In FL(n) let $w \geqq x_r$. If $[w \cap (\bigcup_{i\neq r} x_i)] \cup x_r \geqq w$ then w covers $w \cap (\bigcup_{i \neq r} x_i)$.*

PROOF. First note that $w \neq w \cap (\bigcup_{i\neq r} x_i)$ as $w \geqq x_r$, but $\bigcup_{i\neq r} x_i \ngeqq x_r$. Now suppose that $w \geqq y \geqq w \cap (\bigcup_{i\neq r} x_i)$. If $y \ngeqq x_r$, then $\bigcup_{i\neq r} x_i \geqq y$ and hence $y = w \cap (\bigcup_{i\neq r} x_i)$. If $y \geqq x_r$, then $w \leqq [w \cap (\bigcup_{i\neq r} x_i)] \cup x_r \leqq y \cup x_r = y \leqq w$ and hence $y = w$.

THEOREM 1. *In FL(n), let $\bigcup_{i\neq r} x_i \geqq y$. Then $x_r \cup y$ covers $(x_r \cup y) \cap (\bigcup_{i\neq r} x_i)$.*

PROOF. We set $w = x_r \cup y$ and verify the criteria of the lemma. Since $\bigcup_{i\neq r} x_i \geqq y$ it follows that $(x_r \cup y) \cap (\bigcup_{i\neq r} x_i) \geqq y$ and hence $[(x_r \cup y) \cap (\bigcup_{i\neq r} x_i)] \cup x_r \geqq x_r \cup y$. The theorem now follows.

COROLLARY. *Let FL(3) have generators a, b, and c. If w is any word, then $a \cup (b \cap w)$ covers $[(a \cup (b \cap w)] \cap (b \cup c)$.*

PROOF. $b \cup c \geqq b \geqq b \cap w$.

Note that $a \cup (b \cap w)$ is the form of a typical element in an infinite ascending chain of elements in FL(3) as established in [2], thus this corollary gives an infinite number of pairs of distinct coverings in FL(3).

If $w \geqq x_r$, the criterion of the lemma really states that $w=[w\cap(\bigcup_{i\neq r} x_i)]\cup x_r$. This is certainly a necessary condition for a word w such that $w \geqq x_r$ to cover $w\cap(\bigcup_{i\neq r} x_i)$. Since no element of $FL(n)$ is both meet and join reducible, it follows that if $w \geqq x_r$ and w is meet reducible then w properly contains $x_r\cup[w\cap(\bigcup_{i\neq r} x_i)]$ which, by Theorem 1 does cover $(x_r\cup[w\cap(\bigcup_{i\neq r} x_i)])\cap(\bigcup_{i\neq r} x_i)$. The characterization of those w such that w covers or equals $x_r\cup[w\cap(\bigcup_{i\neq r} x_i)]$ seems difficult. However the following converse for Theorem 1 does hold.

THEOREM 2. *In $FL(n)$, if $w \geqq x_r$ and w covers $w\cap(\bigcup_{i\neq r} x_i)$, then $w=x_r\cup y$ where $\bigcup_{i\neq r} x_i \geqq y$.*

PROOF. The remarks of the preceding paragraph show that we may suppose that w has a canonical form $w_1\cup \cdots \cup w_m$ where w_j is a generator or $w_j=\bigcap_i w_{ij}$. Since $w=[w\cap(\bigcup_{i\neq r} x_i)]\cup x_r$, we can make use of Corollary 2 of Theorem 2 in [1] to infer that for every j, either

$$\text{(a)} \qquad w_j \leqq w \cap \left(\bigcup_{i\neq r} x_i \right) \leqq \bigcup_{i\neq r} x_i,$$

or

$$\text{(b)} \qquad w_j \leqq x_r.$$

Now if (a) holds for all j, then $\bigcup_{i\neq r} x_i \geqq w \geqq x_r$, a contradiction. Hence $w_j \leqq x_r$ for some j. On the other hand, $w \geqq x_r$ implies $w_k \geqq x_r$ for some k. But then the canonicity of the form $w_1\cup \cdots \cup w_m$ implies that $k=j$ (say $=1$) and so $w_1=x_r$. But then canonicity implies that (a) must hold for $j=2, 3, \cdots, m$, or that $y=w_2\cup \cdots \cup w_m \leqq \bigcup_{i\neq r} x_i$, as was to be proved.

REFERENCES

1. P. M. Whitman, *Free lattices*. I, Ann. of Math. vol. 42 (1941) pp. 325–330.
2. ———, *Free lattices*. II, Ann. of Math. vol. 43 (1942) pp. 104–115.

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