

ON THE EXISTENCE OF INVARIANT MEASURES

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Communicated by Walter Rudin, July 3, 1961

Under the same title the outline of a rather involved proof of the existence of a finitely additive measure invariant under a transformation was recently published in this Bulletin [4]. A more general theorem was announced by the author several years ago in [2]. In this note first the original simple proof (unpublished) will be applied to yield a further generalization of the latter result (Theorem 1). Then its thesis will be strengthened under some of the assumptions made in the above publications (Theorem 2). Finally, some fundamental properties of the invariant set function will be established (Theorem 3).

Let S be a class of subsets of a set X . For each element g of a semigroup G , let \bar{g} be an additive mapping of S into itself such that

$$\bar{h}\bar{g}A = \bar{g}(\bar{h}A) \quad \text{for all } g, h \in G, A \in S.$$

If m is any set function on S , then call a set function μ defined on a class of sets $\Sigma \supset S$ a G -perfection of m on Σ if it has the following properties: (i) it is finitely additive; (ii) it is G -invariant i.e. $\mu(\bar{g}A) = \mu(A)$ for all $g \in G$ and $A \in S$; (iii) all its values are between the extreme bounds of those of m ; (iv) $\mu(A)$ is between the extreme bounds of the set of all numbers $m(\bar{g}A)$ with $g \in G$ for every $A \in S$.

THEOREM 1. *If the semigroup G is abelian (or, more generally, left measurable²), then every finitely additive and bounded set function m on S has a G -perfection on S .*

PROOF. It was shown in [3] that every abelian semigroup G is left measurable i.e. that there is a linear functional K on the set B of all bounded real functions on G such that (a) $f \in B, f_h(g) = f(hg)$ for all $g \in G$ implies $K(f_h) = K(f)$ for all $h \in G$, (b) $K(1) = 1$, (c) $K(f) \geq 0$ if $f(g) \geq 0$ for all g in G . Theorem 1 is proved by straightforward verification that the set function μ defined by the formula

$$\mu(A) = K[m(\bar{g}A)], \quad A \in S,$$

is a G -perfection of m on S .

¹ Sponsored by the United States Army under Contract No. DA-11-022-ORD-2059 and United States Air Force under Contract No. AF 18 (600).

² See definition in the proof.

It is easily seen that this theorem already generalizes the above mentioned results. Their strengthening follows from the next theorem.

THEOREM 2. *If S is a ring of sets and if there is a mapping T_g of X into X such that $\bar{g}A = T_g^{-1}A$ for every $g \in G$ and $A \in S$, and G is left measurable, then every set function m satisfying the assumptions of Theorem 1 has a G -perfection on the class $P(X)$ of all subsets of X .*

To prove this theorem, it is enough to show: (A) *Every finitely additive bounded set function m on S can be extended to a like set function \bar{m} on $P(X)$ without changing the bounds of its values.*

In fact, if (A) is true, then replacing m and S in Theorem 1 by \bar{m} and $P(X)$, respectively, yields the thesis of Theorem 2.

PROOF OF (A). The integral with respect to m is a linear functional on the set of all S -simple functions and is there dominated by the sublinear functional $p(f) = \sup\{f(x) : x \in X\} \sup\{|m|(A) : A \in S\}$ where $|m|(A)$ is the total variation of m on A and $f \in B$. Therefore, by a well-known theorem of Banach,³ this integral can be extended to a linear functional $I \leq p$ on B . Then the set function \bar{m} defined by the formula

$$\bar{m}(A) = I(\chi_A) \quad \text{for } A \subset X, \chi_A = \text{characteristic function of } A,$$

is readily seen to have all the properties stated in (A).

THEOREM 3. *Let S be a σ -ring and μ a G -perfection of a finite measure m on S . Then μ is a measure and (a)⁴ $\mu \gg m$ or (b)⁵ $\mu \sim m$ if (and obviously only if) there is some G -invariant finite measure ν on S such that (a') $\nu \gg m$ or (b') $\nu \gg m \gg m_g$ for all $g \in G$, respectively, where $m_g(A) = m(\bar{g}A)$ for all $A \in S, g \in G$.*

PROOF. By the G -invariance of ν and (a'), for every $\epsilon > 0$ there is a number $\delta > 0$ such that $\nu(A) < \delta$ implies

$$0 \leq \inf\{m(\bar{g}A) : g \in G\} \leq \mu(A) \leq \sup\{m(\bar{g}A) : g \in G\} \leq \epsilon.$$

Hence $\lim_n \nu(A_n) = 0$ implies $\lim_n \mu(A_n) = 0$ and μ is a measure.

If $\mu(A) = 0$, then the maximum of ν on the class D of all countable unions of A and of sets of the form $\bar{g}A, g \in G$, is attained at some $B \in D$. Clearly both $\bar{g}B$ and $B \cup \bar{g}B$ are in D and $\nu(B) = \nu(\bar{g}B) = \nu(B \cup \bar{g}B)$ hence $0 = \nu(B \Delta \bar{g}B) = m(B \Delta \bar{g}B)$ and thus $m(\bar{g}B) = m(B)$ for all $g \in G$. However, $\mu(B)$ is between the extreme bounds of the set of

³ [1, p. 29].

⁴ i.e. $\mu(A) = 0$ implies $m(A) = 0$ for all $A \in S$.

⁵ i.e. $\mu \gg m$ and $m \gg \mu$.

all numbers $m(\bar{g}B)$ with $g \in G$. Therefore, $m(B) = \mu(B) = 0$ and, since $A \subset B$, $m(A) = 0$, i.e. (a) holds. The implication (b') \rightarrow (b) readily follows from (a') \rightarrow (a) and from the inclusion of μ between the extreme bounds of the m_g , which completes the proof.

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