A REPRESENTATION OF THE INFINITESIMAL GENERATOR OF A DIFFUSION PROCESS[1]

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0. **Introduction.** Let Ω be a connected locally compact metric space and let $C(\Omega)$ denote, as usual, the Banach space of bounded continuous real functions on Ω. A diffusion process (see [1] for definitions) is a semi-group $\{T_t;\, t>0\}$ of positivity preserving bounded linear transformations on $C(\Omega)$ which is strongly continuous for $t>0$. Such semi-groups are also required to be of local character; i.e., if x vanishes in a neighborhood of a point $\xi\in\Omega$, then

$$Ax(\xi) = \lim_{t\downarrow 0} \frac{T_t x - x}{t}(\xi) = 0.$$

Consider an arbitrary element $x\in C(\Omega)$. If (i) the above limit exists for all η in a neighborhood W of ξ, (ii) the convergence is bounded on this neighborhood, and (iii) Ax is continuous on W, then x is said to be in the local domain of the operator A at ξ. x is said to be in the global domain, $D(A)$, of the operator A whenever $W=\Omega$. Feller (see [1]) has posed the problem of characterizing the operator A. A local representation of such operators will be discussed in this note.

One of the essential properties of the operator A is the maximum property; i.e., $Ax(\xi)\leqq 0$ whenever x is in the local domain of A at ξ and x has a null maximum at ξ. Before discussing the representation of A, a few remarks concerning the denseness in $C(\Omega)$ of the global domain of the operator A are in order. A null point of A is a point $\xi\in\Omega$ such that $Ax(\xi)=x(\xi)=0$ for all x in the local domain of A at ξ. Feller has shown that the set N of null points is a closed set. He has also shown that if x vanishes outside a compact set which does not meet N, then there is a sequence X_λ in the global domain of A such that $X_\lambda\to x$ strongly [1]. Using this result, one can show that $D(A)$ is locally dense in $C(\Omega)$ at each point $\xi\in\Omega-N$; i.e., if (i) $\xi\in\Omega-N$, (ii) W is a neighborhood of ξ such that $\overline{W}\subset\Omega-N$, and (iii) $x\in C(\Omega)$, then x can be approximated uniformly over $\overline{W}$ by an element of $D(A)$.

Another type of point at which the operator A may be degenerate is the absorption point; i.e., ξ is an absorption point if there is a real

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number c such that $Ax(\xi) = cx(\xi)$ for all x in the local domain of A at ξ. It will be assumed throughout this note that ξ is a fixed point of Ω which is neither a null point nor an absorption point. This assumption implies that there is an x in the local domain of A at ξ such that $Ax(\xi) > 0$; V will denote a neighborhood of ξ with compact closure such that $\overline{V} \cap N = \emptyset$ and $Ax > 0$ on $\overline{V}$. If $\eta \in V$, then $D^*(A, \eta)$ will denote the functions in the local domain of A at η restricted to $\overline{V}$. $D^*(A)$ will denote the set of functions obtained by restricting elements of $D(A)$ to $\overline{V}$, and $C(\overline{V})$ will denote the Banach space of continuous real functions on $\overline{V}$. By the above remarks, $D^*(A)$ is dense in $C(\overline{V})$.

1. **Generalized harmonic measures.** It will be assumed in this section that $1 \in D(A)$ and $A1 = 0$. Let U be an open subset of V. The boundary of U will be denoted by U'. A function y in $D^*(A, \eta)$ for all $\eta \in U$ will be called subregular (superregular) on U if $Ay \geqq 0$ ($Ay \leqq 0$) on U. A function is regular on U if it is both subregular and superregular on U. The set P of functions subregular on U has the following properties:

(a) P is a wedge in $C(\overline{V})$; i.e., P is a convex set in $C(\overline{V})$ and $tP \subset P$ for $t \geqq 0$.

(b) $x \in P$ implies $x(\eta) \leqq \sup_{U'} x$ for all $\eta \in U$.

(c) P contains a nonzero element.

The second assertion follows from the fact that $A1 = 0$ and that A possesses the maximum property. These three properties suffice to prove the following theorem.

THEOREM 1. *For each $\eta \in U$, there is a regular Borel measure $p(\eta, U, \cdot)$ defined on the Borel subsets of U' such that*

$$x(\eta) \leqq \int_{U'} x(\sigma) p(\eta, U, d\sigma)$$

for each x subregular on U. Moreover, $p(\eta, U, U') = 1$.

SKETCH OF PROOF. Consider $C(\overline{V}) \times E_1$, the Cartesian product of $C(\overline{V})$ with the set of real numbers. The set E of all pairs (x, α) where $x \in C(\overline{V})$ and $\alpha \geqq \sup_{U'} x$ is a convex body in this product space. The set F of all pairs $(x, x(\eta))$ where x is subregular on U is a convex set in the product space. Moreover, $(\text{Int } E) \cap F = \emptyset$. Using the Eidelheit separation theorem, there is a linear functional y^* on $C(\overline{V})$ such that $x(\eta) \leqq y^*(x)$ for all x subregular on U and $y^*(x) \leqq \sup_{U'} x$ for all $x \in C(\overline{V})$. The Riesz representation theorem can be used to represent the positivity preserving linear functional y^* as a measure on $\overline{V}$. To show that this measure is concentrated on U', it need only be observed that $y^*(z)=0$ for any $z\in C(\overline{V})$ which is zero on U' and strictly positive elsewhere.

A measure of the type described in the preceding theorem will be called a generalized harmonic measure. The integral relative to a generalized harmonic measure will be denoted by $L(\eta, U, \cdot)$.

2. **Representations.** For the time being, it will be assumed that $1\in D(A)$ and $A1=0$. Again U will be an open subset of V. The inequality of the following lemma is the starting point of the representation.

LEMMA 2. *There is an $x\in D^*(A)$ such that $Ax>0$ on $\overline{V}$, $x(\eta) < L(\eta, U, x)$ for all $\eta\in U$, and*

$$\inf_{\overline{U}} (Az/Ax) \leqq \frac{L(\eta, U, z) - z(\eta)}{L(\eta, U, x) - x(\eta)} \leqq \sup_{\overline{U}} (Az/Ax)$$

whenever Az is defined on $\overline{U}$.

SKETCH OF PROOF. One first shows that there is a y (which may depend upon η and U) such that $y(\eta) < L(\eta, U, y)$ and $Ay>0$ on $\overline{U}$ as follows. Suppose the contrary; i.e., $y(\eta)=L(\eta, U, y)$ for all y such that $Ay>0$ on $\overline{U}$. By assumption, there is an x such that $Ax>0$ on $\overline{V}\supset\overline{U}$. Consider any y such that Ay is defined on $\overline{U}$. Each such y can be represented in the form $z-tx$ where $Az>0$ on $\overline{U}$ and t is sufficiently large. It follows that $y(\eta)=L(\eta, U, y)$ for all y for which Ay is defined on $\overline{U}$. But since the class of such functions is dense in $C(\overline{U})$, the evaluation linear functional $y^*(y)=y(\eta)$ and the linear functional $L(\eta, U, \cdot)$ are equal. This, however, is not possible. This proves that there is a y such that $Ay>0$ on $\overline{U}$ and $y(\eta) < L(\eta, U, y)$. A preliminary version of the lemma is now obtained as follows. Consider any z such that Az is defined on $\overline{U}$. For $t\geqq -\inf_{\overline{U}}(Az/Ay)$, $A(z+ty)\geqq 0$ on $\overline{U}$. By Theorem 1, $z(\eta)+ty(\eta)\leqq L(\eta, U, z+ty)$. Rearranging terms and letting t approach $-\inf_{\overline{U}}(Az/Ay)$ results in the left inequality (with x replaced by y). The other inequality is proved similarly. Having proved the inequality with x replaced by y, it follows that $z(\eta) < L(\eta, U, z)$ for any z such that $Az>0$ on $\overline{V}$ and that y, which may depend upon η and U, may be replaced by any such z.

In passing it is worth noting that the preceding inequality can be used to show that every generalized second order differential operator on $C(\Omega)$ as herein considered has a closed extension. The following theorem is an obvious consequence of Lemma 2.

THEOREM 3. *There is an* $x \in D^*(A, \xi)$ *such that for each* $\eta \in V$ *and each* $z \in D^*(A, \eta)$

$$Az(\eta) = Ax(\eta) \lim_{U \downarrow \{\eta\}} \frac{L(\eta, U, z) - z(\eta)}{L(\eta, U, x) - x(\eta)} \cdot$$

The requirement that $1 \in D(A)$ and $A1 = 0$ can now be removed.

THEOREM 4. *There is a neighborhood* W *of* ξ, *a function* $x \in D^*(A, \xi)$ *with* $x > 0$ *on* $\overline{W}$, *and a function* $y \in D^*(A, \xi)$ *with* $xAy - yAx > 0$ *on* $\overline{W}$ *such that for each* $\eta \in W$ *and each* $z \in D^*(A, \eta)$

$$Az(\eta) = \frac{z(\eta)}{x(\eta)} Ax(\eta) + \frac{x(\eta)Ay(\eta) - y(\eta)Ax(\eta)}{x(\eta)} \lim_{U \downarrow \{\eta\}} \frac{L(\eta, U, z/x) - z(\eta)/x(\eta)}{L(\eta, U, y/x) - y(\eta)/x(\eta)} \cdot$$

SKETCH OF PROOF. Since $D(A)$ is locally dense at ξ, there is an $x \in D^*(A, \xi)$ such that $x(\xi) > 0$. Choose a neighborhood $W_1 \subset V$ of ξ such that $x > 0$ on $\overline{W}_1$. Since ξ is neither a null point nor an absorption point, there is a $y \in D^*(A, \xi)$ such that $x(\xi)Ay(\xi) - y(\xi)Ax(\xi) > 0$. Choose a neighborhood W of ξ such that $W \subset W_1$ and $xAy - yAx > 0$ on $\overline{W}$. After restricting all functions to $\overline{W}$, one defines an operator B on quotients of the form z/x, where $z \in D^*(A, \eta)$ and $\eta \in W$, by the equation $B(zx^{-1}) = Az - zx^{-1}Ax$. This operator has the essential properties used to obtain the representation of the previous theorem.

REFERENCE

1. W. Feller, *The general diffusion operator and positivity preserving semi-groups in one dimension*, Ann. of Math. vol. 60 (1954) pp. 417–436.

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