A SIMPLE TRIANGULATION METHOD FOR SMOOTH MANIFOLDS[1]

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We first triangulate a compact closed m-manifold M^m of differentiability class C^r $(r>1)$ in a euclidean space $E^\nu = E^{m+n}$. The method is simpler than earlier methods (see References) and is applicable to a wider class of spaces (see (F) and (G) below).

(A) For a given $\eta>0$, let $(a_1, \cdots, a_\mu)$ be a set of distinct points on M^m such that each point of M^m is at distance $<\eta$ from at least one point a_i.

Let d be the euclidean distance function in E^ν. For each $k \in (1, \cdots, \mu)$, let

$$(1) \qquad \bar{c}_k^\nu = \left\{ x \in E^\nu \,\middle|\, d(a_k, x) \leq d(a_i, x), \quad i = 1, \cdots, \mu \right\},$$

$$(2) \qquad \bar{\gamma}_k^m = M^m \cap \bar{c}_k^\nu = \left\{ x \in M^m \,\middle|\, d(a_k, x) \leq d(a_i, x), \quad i = 1, \cdots, \mu \right\}.$$

THEOREM. *For each $p \in M^m$, let $\bar{\gamma}(p)$ be the intersection of all the sets $\bar{\gamma}_k^m$ containing p. If η is small enough, $\{\bar{\gamma}\} = \{\bar{\gamma}(p) \,|\, p \in M^m\}$ is a subdivision of M^m into the closed cells of a complex.*

PROOF. Note first that if $i \neq k$, $d(a_k, x) = d(a_i, x)$ defines the normal bisecting $(\nu-1)$-plane $L_{ki}^{\nu-1}$ of the segment $a_k a_i$, and $d(a_k, x) < d(a_i, x)$ defines the half-space H_{ki}^ν of E^ν bounded by $L_{ki}^{\nu-1}$ and containing a_k. Thus $\bar{c}_k^\nu$ is the closure of the open convex polyhedral ν-cell

$$(3) \qquad c_k^\nu = \bigcap_{i \neq k} H_{ki}^\nu = \left\{ x \in E^\nu \,\middle|\, d(a_k, x) < d(a_i, x), \, i \neq k \right\},$$

which may be of infinite diameter.

(B) *The set $\bar{\gamma}_k^m = \bar{c}_k^\nu \cap M^m$ is on the interior $B^\nu(a_k, \eta)$ of the sphere $S^{\nu-1}(a_k, \eta)$ of radius η about a_k.*

For, by (A), each point of $M^m - \bar{B}^\nu(a_k, \eta)$ is closer to some $a_i \neq a_k$ than to a_k, and c_k^ν is the set of all points which are closer to a_k than to any $a_i \neq a_k$.

The fact that M^m is compact and of class C^2 implies that there exists a number $\rho>0$ so small that no $(\nu-1)$-sphere of radius ρ tangent to M^m encloses a point of M^m. The cell c_k^ν therefore contains all points at distances $\leq \rho$ from a_k on the normal n-plane $N^n(a_k)$ to M^m at a_k, since each such point is closer to a_k than to any $a_i \neq a_k$.

(C) Hence, if $L_{ki}^{\nu-1}$ (defined above) intersects $\bar{\gamma}_k^m$, then $L_{ki}^{\nu-1} \cap N^n(a_k)$ is either vacuous or at distance $>\rho$ from a_k.

(D) We take $\eta < \rho/2$ and so small that each intersection $\sigma^m(a_k, \eta)$ $= M^m \cap B^\nu(a_k, \eta)$ is an m-cell, each secant of which, hence each tangent line, makes an angle $\leq \pi/4$ with the tangent plane $T^m(a_k)$ at a_k. The projection $\bar{\sigma}^m \to T^m(a_k)$ is a homeomorphism.

A closed $(n+1)$-dimensional half-plane $\bar{H}^{n+1}$, with $N^n(a_k)$ for boundary, intersects $T^m(a_k)$ in a closed half-line $\tau^1 = T^m(a_k) \cap \bar{H}^{n+1}$ from a_k, and it intersects $\bar{\sigma}^m(a_k, \eta)$ in a *radial arc* κ, whose projection on $T^m(a_k)$ is a segment of τ^1.

(E) *If κ is a radial arc of $\sigma^m(a_k, \eta)$ which intersects the plane $L_{ki}^{\nu-1}$ of statement* (C) *on $B^\nu(a^\kappa, \eta)$, then $\kappa \cap L_{ki}^{\nu-1}$ is a single point, at which κ crosses $L_{ki}^{\nu-1}$.*

For otherwise, $L_{ki}^{\nu-1}$ contains a line L^1 which is a tangent or a secant to κ. Since L^1 lies in $\bar{H}^{n+1}$, makes an angle less than $\pi/4$ with τ^1, and passes, by (B), within η of a_k, it intersects $N^n(a_k)$ within distance $2^{1/2}\eta$ of a_k. But, $L^1 \cap N^n(a_k) \subset L_{ki}^{\nu-1} \cap N^n(a_k)$ must be at distance $> \rho > 2\eta$ from a_k, by (C) and (D). This contradiction proves (E).

It now follows easily, using (B), that each radial segment of $\sigma^m(a_k, \eta)$ intersects $\bar{c}_k^\nu - c_k^\nu$ in a single point, so that $\gamma_k^m = \sigma^m(a_k, \eta) \cap c_k^\nu$ is an m-cell. The proof of the theorem is easy to complete. The complex $\{\gamma\}$ can be simplicially subdivided into a C^r-triangulation of M^m.

(F) It is possible to extend this method to a noncompact M^m and to a manifold with boundary. In the noncompact case, $(a_1, \cdots, a_\mu)$ is replaced by a countable set without limit point satisfying a density requirement as in (A), where η is replaced by a suitable positive function on M^m.

(G) The method can be modified to treat the case where M^m is of class C^1 or is subject only to an appropriate Lipschitz condition. Locally triangulable structures made up of differentiable or Lipschitz manifolds, suitably restricted, can also be thus triangulated.

We hope to publish such extensions elsewhere, along with theorems regarding isotopic deformations carrying the resulting complexes into polyhedra.

References

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