AREA OF DISCONTINUOUS SURFACES

BY CASPER GOFFMAN[1]

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1. A general theory of surface area, [**1**; **2**], exists for the non-parametric case. Thus, area is defined for all measurable f on the unit square $Q=I\times J$. The area functional is lower semi-continuous with respect to almost everywhere convergence and agrees with the Lebesgue area for continuous f. On the other hand, for continuous parametric mappings T of the closed unit square Q into euclidean 3-space E_3, Lebesgue area is not lower semi-continuous with respect to almost everywhere convergence nor even, as C. J. Neugebauer has shown, [**3**], with respect to pointwise convergence.

It thus appears that a theory of parametric surface area must be restricted to surfaces which cannot deviate too far from the ones given by continuous mappings. In this paper, we develop the beginnings of a theory for a class of surfaces which we call linearly continuous.

2. Let f be a real function defined on Q and, for every u, let f_u be defined by $f_u(v)=f(u, v)$ and let f_v be defined similarly. Then f is linearly continuous if f_u is continuous for almost all u and f_v is continuous for almost all v. A mapping $T: x=x(u, v),\ y=y(u, v),\ z=z(u, v)$ of Q into E_3 is linearly continuous if x, y, z are linearly continuous.

A sequence $\{f_n\}$ of functions converges linearly to a function f if $(f_n)_u$ converges uniformly to f_u for almost all u, and $(f_n)_v$ converges uniformly to f_v for almost all v. A sequence $T_n: x=x_n(u, v), y=y_n(u, v), z=z_n(u, v)$ converges linearly to a mapping $T: x=x(u, v), y=y(u, v), z=z(u, v)$ if $\{x_n\}, \{y_n\}, \{z_n\}$ converge linearly to x, y, z, respectively.

Let P be the set of quasi linear mappings from Q into E_3. For $p, q\in Q$ let

$$\begin{aligned} d(p, q) = \inf[k\colon &\text{there are sets } A_k \subset I,\ B_k \subset J, \\ &m(A_k) > 1 - k,\ m(B_k) > 1 - k, \text{ and } |p(u, v) - q(u, v)| < k \\ &\text{on } (A_k \times J) \cup (I \times B_k)]. \end{aligned}$$

It is easy to verify that P is a metric space and that $\{p_n\}$ converges to p in this space if and only if it converges linearly. Let E be the elementary area functional on P. It is not hard to prove

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THEOREM 1. *E is lower semi-continuous on P. In other words, if $\{p_n\}$ converges linearly to p then* $\liminf E(p_n) \geqq E(p)$.

By the Fréchet extension theorem, E is extended to a lower semi-continuous functional Φ on the completion $\mathfrak{L}$ of P.

THEOREM 2. *The completion $\mathfrak{L}$ of P is the space of linearly continuous mappings with the metric corresponding (as above) to linear convergence.*

3. It is obvious that for every continuous mapping T, $A(T) \geqq \Phi(T)$ where $A(T)$ is the Lebesgue area. The inverse inequality holds so that the functional Φ constitutes a legitimate extension of Lebesgue area to substantially wider class of mappings than the continuous ones. We outline the proof.

For a continuous $T: x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, the lower area $V(T)$ is defined as follows:

Let $T_1: y=y(u, v)$, $z=z(u, v)$, $T_2: x=x(u, v)$, $z=z(u, v)$, and $T_3: x=x(u, v)$, $y=y(u, v)$ be the associated flat mappings. For every simple polygonal region P in Q^0, let

$$v_1(P) = \int |O(\xi, T_1P^*)|,$$

where the integration is over the yz plane, and $O(\xi, T_1P^*)$ is the topological index of T_1P^* at ξ (A^0 and A^* are the interior and boundary, respectively, of a set A). Define $v_2(P)$ and $v_3(P)$, similarly, and let

$$v(P) = [v_1(P)^2 + v_2(P)^2 + v_3(P)^2]^{1/2}.$$

Let $\pi = (P_1, \cdots, P_n)$ be a finite set of pair-wise disjoint simple polygonal regions in Q^0 and

$$v(\pi) = \sum_{i=1}^{n} v(P_k).$$

Finally, let

$$V(T) = \sup[v(\pi): \pi].$$

Cesari has shown (e.g. [**4**]) that $A(T) = V(T)$ for every continuous T.

The distance between 2 sets A and B is defined by

$$d(A, B) = \sup[d(x, B): x \in A] + \sup[d(y, A): y \in B].$$

With this metric, the set α of simple polygonal regions is a separable metric space. Let $\beta \subset \alpha$ be dense in α and

$$V_\beta = \sup[v(\pi): \pi \subset \beta].$$

LEMMA 1. $V_\beta(T) = V(T)$.

Now, let $\{T_n\}$ be a sequence of continuous mappings which converges linearly to a continuous mapping T. Let γ be the set of simple polygonal regions whose boundaries consist of line segments parallel to the coordinate axes for which T and T_n, $n=1, 2, \cdots$ are continuous and on each of which $\{T_n\}$ converges uniformly to T. For each $\pi \subset \gamma$, $\liminf v(\pi, T_n) \geqq v(\pi, T)$. Since γ is dense in α, it follows that $\liminf V(T_n) \geqq V(T)$. This proves

THEOREM 3. *$A(T)$ is lower semi-continuous with respect to linear convergence on the set of continuous mappings.*

COROLLARY 1. *$A(T) = \Phi(T)$ for every continuous T.*

PROOF. For every sequence $\{p_n\}$ of quasi-linear mappings converging linearly to T, $\liminf E(P_n) \geqq A(T)$. Choose $\{p_n\}$ so that $\lim E(p_n) = \Phi(T)$. Then $A(T) \leqq \Phi(T)$,

4. A set S will be called negligible if $S \subset Z_1 \times Z_2$ where Z_1 and Z_2 have linear measure zero. Kolmogoroff's principle holds in the following form.

THEOREM 4. *If T_1 and T_2 are linearly continuous mappings from Q into E_3 and if for every pair of points ξ, η not belonging to a negligible set*

$$|T_1\xi - T_1\eta| \leqq |T_2\xi - T_2\eta|,$$

then $\Phi(T_1) \leqq \Phi(T_2)$.

5. A real function f on Q is BVC if for almost all u and almost all v, f_u and f_v are equivalent to functions of bounded variation and the corresponding variation functions are summable. f is ACE if for almost all u and almost all v, f_u and f_v are equivalent to absolutely continuous functions.

For functions which are BVT and ACT it is a simple known fact that the integral means commute with the partial derivatives. This also holds almost everywhere for functions which are BVC and ACE. Using this fact and the fact, [**5**], that if f is BVC and linearly continuous then the integral means of f converge linearly to f, the proof of the following generalization of a theorem of Morrey, [**4**], may be obtained in somewhat standard fashion. The generalization is in two directions. Instead of holding only for conjugate Lebesgue spaces, the theorem holds for conjugate Köthe spaces, [**6**; **7**], and the theorem holds for linearly continuous mappings rather than just for continuous ones.

THEOREM 5. *If the functions x, y, z of a linearly continuous T are BVC and ACE and if the pairs of partial derivatives* (x_u, y_v), (x_v, y_u), (x_u, z_v), (x_v, z_u), (y_u, z_v), (y_v, z_u) *belong to conjugate Köthe spaces, the area* $\Phi(T)$ *is given by the formula*

$$\Phi(T) = \int J\, dudv$$

where $J = [J_1^2 + J_2^2 + J_3^2]^{1/2}$ *and* J_1, J_2, J_3 *are the jacobians of* T_1, T_2, T_3, *respectively.*

6. We define an equivalence relation for linearly continuous mappings. T is equivalent to $T'(T \approx T')$ if there are sequences $\{p_n\}$ and $\{q_n\}$ of quasi linear mappings such that, for every n, $p_n \approx q_n$ in the Lebesgue sense and $\{p_n\}$ converges linearly to T, $\{q_n\}$ converges linearly to T'.

The following simple facts hold:

(a) The relation " $\approx$ " has the properties of an equivalence relation.

(b) If T and T' are continuous and Fréchet equivalent then $T \approx T'$.

(c) If $T \approx T'$ then $\Phi(T) = \Phi(T')$.

We refer to an equivalence class as a surface and to its elements as representations.

D mappings, the Dirichlet integral, and almost conformal mappings are defined as for the continuous case, [4], with BVT and ACT replaced by BVC and ACE.

We say that a mapping T is simple if there is a negligible set S such that $\xi \in Q - S$, $\eta \in Q - S$, $\xi \neq \eta$ implies $T(\xi) \neq T(\eta)$.

The following holds:

THEOREM 6. *If T' is a linearly continuous simple mapping and* $\Phi(T') < \infty$, *the surface given by T' has a representation T, with jacobian J, such that*

$$\Phi(T') = \Phi(T) = \int J\, dudv.$$

COROLLARY. *Every linearly continuous nonparametric surface of finite area has a parametric representation T, with jacobian J, such that*

$$\Phi(T) = \int J\, dudv$$

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Purdue University