APPROXIMATION OF SMOOTH FUNCTIONS

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Theorems of Jackson and S. Bernstein about the approximation of smooth functions are usually interpreted in the way that all functions with a prescribed degree of smoothness have a definite degree of approximation. They can be viewed in another way, which reveals their susceptibility to generalization.

Let $\omega(h)$ be an increasing continuous subadditive function defined for $h \geqq 0$ with $\omega(0)=0$, A a compact metric space with infinitely many points. By C_1^{ω} we denote the set of all real valued functions f on A with $|f(x)| \leqq 1$, $|f(x)-f(x')| \leqq \omega(h)$, $h=\rho(x, x')$. If A is a q-dimensional cube, p a natural number and $0<\alpha \leqq 1$, we denote by $C_1^{p+\alpha}$ the set of all functions on A with continuous partial derivatives of orders not exceeding p and bounded by 1, and with the derivatives of order p satisfying a Lipschitz condition of order α and with coefficient 1. Let $G=\{g_n\}$ be a sequence of continuous functions on A. Then, with some norm, for example the uniform norm on A,

$$E_n(f) = E_n^G(f) = \inf \left\| f - \sum_{i=1}^{n} a_i g_i \right\|$$

is the degree of approximation of f by linear combinations of $g_1, \cdots, g_n$; and

$$\mathcal{E}_n(W) = \sup_{f \in W} E_n(f)$$

is the degree of approximation of a class W.

The theorems of Jackson and Bernstein state that for periodic $f \in C_1^{p+\alpha}$, and the trigonometric approximation, $E_n(f)$ has the exact order $n^{-(p+\alpha)/q}$; exceptions occur only if f has a higher degree of smoothness. We regard this as a statement about a certain massivity of $C_1^{p+\alpha}$, which prevents better approximation by linear combinations of only n functions. One can hope that an estimate of $\mathcal{E}_n(C_1^{p+\alpha})$ from below can be given for an arbitrary system G, and that the trigonometric system is close to the best possible. That this is true, is shown by the following results:

THEOREM 1. *Let A be a compact metric space, and $\delta=\delta(n)$ the largest*

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number such that there exist n points of A with mutual distances $\geqq \delta$. *Then for each G,*

$$\mathcal{E}_n(C_1^{\omega}) \geqq \frac{1}{2}\,\omega(\delta(n+1)).$$

This cannot be essentially improved, for there exists a G with $\mathcal{E}_n(C_1^{\omega}) \leqq \omega(\delta_1)$, if A can be covered by n open balls of radius $\delta_1 = \delta_1(n)$.

THEOREM 2. *If A is a q-dimensional cube, then for some constant B, and each G, in the uniform and the* L^1 *norm*

$$\mathcal{E}_n(C_1^{p+\alpha}) \geqq Bn^{-(p+\alpha)/q}, \qquad p = 0, 1, \cdots; 0 < \alpha < 1. \tag{1}$$

From these and similar theorems one can obtain by a method of condensation of singularities:

THEOREM 3. *If A is as in Theorem* 2, $p=0, 1, \cdots$ *and* $0<\alpha<1$, *then there exists a constant B such that for each system G one can find a function* $f_0 \in C_1^{p+\alpha}$ *such that, in the uniform and the* L^1 *norm,*

$$E_n(f_0) \geqq Bn^{-(p+\alpha)/q}$$

for an infinite number of values of n.

THEOREM 4. *Let* A_ρ *be the ellipse with focii* -1, $+1$ *and the sum of the half-axes* 2ρ. *For each G and each sequence* $\epsilon_n \to 0$ *there exists a function* $f_0(z)$, *analytic inside* A_ρ, *with* $|f_0(z)| \leqq 1$ *such that the degree of approximation of* f_0 *on* $(-1, +1)$ *satisfies, in the* L^1 *norm,*

$$E_n(f_0) \geqq \epsilon_n \rho^{-n} \tag{2}$$

for infinitely many n.

Similar and more general results were recently obtained by A. G. Vituškin [1]. However, his lower bounds (for $s=n+1$, $m=0$ in [1]) are of orders $(n \log n)^{-(p+\alpha)/2q}$ and n^{-an}, $a>0$, for the problems of types (1), and (2), respectively.

REFERENCE

1. A. G. Vituškin, *Best approximation of differentiable and analytic functions*, Dokl. Akad. Nauk SSSR vol. 119 (1958) pp. 418–420.

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