SOME PROPERTIES OF CHARACTERS OF FINITE SOLVABLE GROUPS

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The purpose of this note is to announce two results on properties of the characters of a finite solvable group G. The first is a necessary and sufficient condition on the irreducible characters of G in order that its Sylow subgroups be abelian. The second is a necessary condition on the irreducible characters of G in order that certain p-subgroups of G be abelian. These results confirm certain assertions in a conjecture by Brauer in [1]. In order to formulate the results, some definitions from modular representation theory are needed. For the motivation behind these definitions, see [2].

Let G be a finite group of order g, F an algebraic number field of finite degree such that the irreducible representations of G in F are absolutely irreducible. If p is a fixed rational prime, there is a grouping of the absolutely irreducible characters $\chi_1, \chi_2, \cdots, \chi_k$ of G into disjoint sets called the blocks of G (for the prime p). This grouping can be made in the following way: let $\mathfrak{p}$ be a prime ideal divisor of p in F. Then two irreducible characters χ_i and χ_j are in the same block B_t of G if and only if

$$\frac{g}{n(\sigma)} \frac{\chi_i(\sigma)}{x_i} \equiv \frac{g}{n(\sigma)} \frac{\chi_j(\sigma)}{x_j} \pmod{\mathfrak{p}}$$

for all σ in G. Here $n(\sigma)$ is the order of the normalizer of σ in G, x_i the degree of χ_i, x_j the degree of χ_j. Since the numbers occurring in the above congruence can be computed from the character table of G, the blocks of G can therefore be determined once the character table of G is known.

THEOREM 1. *Let G be a finite solvable group, B_1 the block containing the principal or 1-character of G. Then a necessary and sufficient condition for the Sylow p-subgroups of G to be abelian is that every character in B_1 has degree relatively prime to p.*

Since our two results are related, we shall state the second result before indicating their proofs. Let p^a be the highest power of p dividing the order of the finite group G (G not necessarily solvable). The defect of a block B_t of G is the smallest non-negative integer d such that p^{a-d} divides the degree x_i of every character χ_i in B_t. If the exact power of p dividing the degree of a character χ_i in B_t is p^{a-d+e}, where $e \geqq 0$, we define the height of χ_i to be e. In [2] there is associated to each block B_t its defect group D, a p-subgroup of G determined uniquely up to conjugate subgroups in G. D has order p^d, where d is the defect of B_t. The conjecture by Brauer is that the defect group D of a block B_t is abelian if and only if every character in B_t has height 0. Theorem 1 is a special case of the conjecture, since the defect groups of B_1 are the Sylow p-subgroups of G.

THEOREM 2. *Let G be a finite solvable group, B_t a block of G with D as defect group. If the center of D has index p^c in D, then every character in B_t has height less than or equal to c. In particular, if D is abelian, then every character in B_t has height* 0.

The necessity of the condition in Theorem 1 is included in Theorem 2. The proofs of both theorems use induction on the order of G. In order to carry through the induction, the following lemma is needed: let H be a normal subgroup of prime index of a group G, G not necessarily solvable. If χ_i is an irreducible character of G in a block with D as defect group, then some irreducible constituent of the restriction $\chi_i|H$ of χ_i to H is in a block of H with $D\cap H$ as defect group. Here $D\cap H\neq D$ only in the case $(G: H)=p$ and the irreducible constituents of $\chi_i|H$ are in one block of H. Theorem 2 follows by a direct application of this lemma except in the case where $D\cap H\neq D$ and $\chi_i|H$ is reducible. However, it is then possible to show that the center of D lies in H and induction will work.

The proof of the sufficiency of the condition in Theorem 1 is rather lengthy. The bare outline of the proof is as follows. By induction we can reduce the problem first to the case where every maximal normal subgroup of G has index p, secondly to the case where every minimal normal subgroup of G has order a p-power. These restrictions on the maximal and minimal normal subgroups of G imply that the minimal normal subgroups of G have order p. If Theorem 1 were not true, it would then be possible to construct an irreducible character of a suitable subgroup M of G for which Theorem 2 would be false. The subgroup M is the normalizer in G of a fixed Sylow p-subgroup of H, where H is a maximal normal subgroup of G.

REFERENCES

1. R. Brauer, *Number theoretical investigations on groups of finite order*, Proceedings of the International Symposium on Algebraic Number Theory, Tokyo, 1956, pp. 55–62.

2. ——— *Zur Darstellungstheorie der Gruppen endlicher Ordnung*, Math. Z. vol. 63 (1956) pp. 406–444.

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