A REDUCTION OF THE SCHOENFLIES EXTENSION PROBLEM

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1. In his study of the Schoenflies extension problem Mazur, [1], includes the hypothesis that the given mapping ϕ be "semilinear" in a neighborhood of some point in the shell domain of ϕ. In Mazur's paper the problem of removing this hypothesis is unsolved. We shall show how to remove this hypothesis, leaving the "shell hypothesis." In the process of doing this we establish two fundamental lemmas concerning the equivalence of extension problems.

Let E and $\mathcal{E}$ be euclidean n-spaces, $n>1$, with rectangular coordinates (x) and (y) respectively. Let (x) and (y) be represented by vectors $\mathbf{x}$ and $\boldsymbol{y}$ with components $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ respectively. Let $\|\mathbf{x}\|$ be the distance of the point $\mathbf{x}$ from the origin. Let I be the mapping of E onto $\mathcal{E}$ under which $\mathbf{x}=\boldsymbol{y}$. Let S_{n-1} be an $(n-1)$-sphere of unit radius in E with center at the origin. Let $\mathring{J}S_{n-1}$ be the interior of S_{n-1}. With $0<a<1$, introduce the n-shell

$$\sigma_a = \{\mathbf{x} \mid 1 - a < \|\mathbf{x}\| < 1 + a\}. \tag{1}$$

As applied to a mapping of an open subset of a euclidean space or a differentiable manifold into a similar space, a *C^m-diffeomorphism* shall have its usual meaning when $m>0$, but when $m=0$ shall be merely a *homeomorphism.*

In this sense let

$$\phi: \sigma_a \rightarrow \mathcal{E} \tag{2}$$

be a C^m-diffeomorphism of σ_a onto a subset of $\mathcal{E}$. The set $\phi(\sigma_a)$ is open in $\mathcal{E}$ and affords a neighborhood of the manifold $\phi(S_{n-1}) = \mathfrak{M}_{n-1}$. Let $\mathring{J}\mathfrak{M}_{n-1}$ be the interior of $\mathfrak{M}_{n-1}$. We assume that ϕ carries points of σ_a which are interior to S_{n-1} into points which are interior to $\mathfrak{M}_{n-1}$. We term these conditions on ϕ the *shell hypothesis.*

THEOREM 1. (i) *If N is a sufficiently small neighborhood of S_{n-1} there exists an extension Λ_ϕ of $\phi|N$ which is a homeomorphism of $N \cup \mathring{J}S_{n-1}$ onto $\phi(N) \cup \mathring{J}\mathfrak{M}_{n-1}$.*

(ii) *If $m>0$ and if Z is an arbitrary point interior to S_{n-1}, then Λ_ϕ and N may be chosen so that Λ_ϕ is, in addition, a C^m-diffeomorphism of $N \cup \mathring{J}(S_{n-1}) - Z$ into $\mathcal{E}$.*

In the differentiable case $(m>0)$, the theorem has been proved by the author in [2]. In the topological case $(m=0)$ we shall give a proof of (i) by transforming the problem into an equivalent problem solvable by the methods of Mazur in [1], or by the subsequent methods of the author in [2].

Given the C^m-diffeomorphism ϕ of Theorem 1, we term a mapping Λ_ϕ which satisfies Theorem 1 a solution of the Schoenflies extension problem $[\phi, \sigma_a]$. Let f be a C^m-diffeomorphism of the set

$$\phi(\sigma_a) \cup \mathring{J}\phi(S_{n-1}) = \Omega \tag{3}$$

into $\mathcal{E}$. It must be remembered that when $m=0$, a C^m-diffeomorphism is merely a homeomorphism.

LEMMA 1. (i) *If ϕ is a C^m-diffeomorphism of σ_a into $\mathcal{E}$ satisfying the shell hypothesis, $m \geqq 0$, then $f\phi$ is a C^m-diffeomorphism of σ_a into $\mathcal{E}$ satisfying the shell hypothesis.*

(ii) *A necessary and sufficient condition that there exist a solution Λ_ϕ of the problem $[\phi, \sigma_a]$ is that there exist a solution $\Lambda_{f\phi}$ of the problem $[f\phi, \sigma_a]$.*

PROOF OF (i). It is clear that $f\phi$ is a C^m-diffeomorphism of σ_a into $\mathcal{E}$. Set $f(\phi(S_{n-1})) = \mathcal{L}_{n-1}$. We must show that $f\phi$ carries a point x of σ_a interior to S_{n-1} into a point interior to $\mathcal{L}_{n-1}$.

This will follow once we have proved the relation

$$f(\mathring{J}\mathfrak{M}_{n-1}) = \mathring{J}\mathcal{L}_{n-1}. \tag{4}$$

Now $\mathring{J}\mathfrak{M}_{n-1}$ is the unique, bounded, maximal connected subset of $\mathcal{E}$ which does not meet $\mathfrak{M}_{n-1}$. Set $f(\mathring{J}\mathfrak{M}_{n-1}) = W$. The image set W is bounded and connected. Making use of the fact that f is locally a C^m-diffeomorphism of open subsets of $\mathcal{E}$ onto open subsets of $\mathcal{E}$, one can verify the definition of f^{-1} along all paths in $\mathcal{E}$ that start with points of W and do not meet $\mathcal{L}_{n-1}$. Since the image under f^{-1} remains in $\mathring{J}\mathfrak{M}_{n-1}$ these paths are in W. Hence W is a maximal, connected subset of $\mathcal{E}$ that does not meet $\mathcal{L}_{n-1}$. Since W is bounded $W = \mathring{J}\mathcal{L}_{n-1}$.

Thus (4) holds. A point x of σ_a interior to S_{n-1} has by hypothesis an image $\phi(\mathrm{x})$ interior to $\mathfrak{M}_{n-1}$, and by (4) $f(\phi(\mathrm{x}))$ is interior to $\mathcal{L}_{n-1}$. This establishes (i).

PROOF OF (ii). Suppose that Λ_ϕ is a solution of problem $[\phi, \sigma_a]$. I say that $f\Lambda_\phi$ is a solution of problem $[f\phi, \sigma_a]$. For $f\Lambda_\phi$ is an extension of $(f\phi)|N$ since $f(\Lambda_\phi(\mathrm{x})) = f(\phi(x)) = (f\phi)(\mathrm{x})$, $[\mathrm{x} \in N]$. Moreover $f\Lambda_\phi$ is a homeomorphism of $N \cup \mathring{J}(S_{n-1})$ onto

$$(f\Lambda_\phi)(N \cup \mathring{J}S_{n-1}) = f(\phi(N)) \cup f(\mathring{J}\mathfrak{M}_{n-1}) = (f\phi)(N) \cup \mathring{J}\mathcal{L}_{n-1} \tag{5}$$

in accord with (4). Moreover, if $m>0$, $f\Lambda_\phi$ is a C^m-diffeomorphism of $N \cup \mathring{J}S_{n-1} - Z$ into $\mathcal{E}$.

Conversely if $\Lambda_{f\phi}$ is a solution of problem $[f\phi, \sigma_a]$, defined over $N \cup \mathring{J}(S_{n-1})$, then $f^{-1}\Lambda_{f\phi}$ is a solution of problem $[\phi, \sigma_a]$, defined over $N \cup \mathring{J}S_{n-1}$. One makes use of the relation, derived from (4),

$$\mathring{J}\mathfrak{M}_{n-1} = f^{-1}(\mathring{J}\mathfrak{L}_{n-1}).$$

THE HYPOTHESIS A. *We say that the given mapping ϕ satisfies hypothesis* A *if ϕ reduces to I in some neighborhood X, relative to σ_a, of some point Q of S_{n-1}.*

LEMMA 2. *Corresponding to a given problem $[\phi, \sigma_a]$ of index $m \geqq 0$, for suitable definition of a C^m-diffeomorphism f of the set Ω into $\mathcal{E}$ the C^m-diffeomorphism $f\phi$ of σ_a into $\mathcal{E}$ satisfies Hypothesis* A.

PROOF OF LEMMA 2. Let Q be an arbitrary fixed point of S_{n-1}. For the purposes of this proof suppose that $\phi(Q)$ is the origin in $\mathcal{E}$. Let $\mathcal{K}_\rho$ be an open n-ball in $\mathcal{E}$ with center at the origin and radius ρ. In particular let $\mathcal{K}_{2b}$ have a radius so small that $\mathcal{K}_{2b} \subset \phi(\sigma_a)$. Set

$$\phi^{-1}(\mathcal{K}_b) = X_b, \qquad \phi^{-1}(\mathcal{K}_{2b}) = X_{2b}$$

and note that $X_{2b} \subset \sigma_a$. Let $t \rightarrow \lambda(t)$ be a C^∞-diffeomorphism of the open interval $(0, 2b)$ onto the positive t-axis such that $\lambda(t) = t$ for $t \in (0, b)$. Such a mapping exists.

Let μ be the C^∞-diffeomorphism of $\mathcal{K}_{2b}$ onto $\mathcal{E}$ such that the point $y \in \mathcal{K}_{2b}$ has the image $\lambda(\|y\|)y/\|y\|$ when $\|y\| \neq 0$ and the image O when $y = O$. Set

$$\psi(x) = \mu(\phi(x)) \qquad (x \in X_{2b}), \tag{7}$$

and note that $\psi(x) = \phi(x)$ for $x \in X_b$ and that $\psi(X_{2b}) = \mathcal{E}$. We can accordingly define a C^m-diffeomorphism f over Ω by setting

$$f(y) = I(\psi^{-1}(y)) \qquad (y \in \Omega). \tag{8}$$

We have

$$f(\phi(x)) = I(x) \qquad (x \in X_b). \tag{9}$$

Thus $f\phi$ satisfies Hypothesis A.

Theorem 1 (i) follows from the two lemmas taking account of the known fact that the problem $[f\phi, \sigma_a]$ admits a solution even when $m = 0$, since $f\phi$ satisfies, not only the shell hypothesis, but also hypothesis A.

REFERENCES

1. Barry Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 59–65.

2. Marston Morse, *Differentiable mappings in the Schoenflies theorem*, Compositio Math. vol. 14 (1959) pp. 83–151.

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