REMARKS ON AFFINE SEMIGROUPS[1]

BY A. D. WALLACE

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A *semigroup* is a nonvoid Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition. In what follows S will denote one such and it will be assumed that S is *compact*. It thus entails no loss of generality to suppose that S is contained in a locally convex linear topological space $\mathfrak{X}$, but no particular imbedding is assumed. For general notions about semigroups we refer to [3] and for information concerning linear spaces to [2].

It has been known for some time [3] that if $\mathfrak{X}$ is finite dimensional, if S is convex (recall that S is compact) and if S has a unit (always denoted by u) then the maximal subgroup, H_u, which contains u is a subset of the boundary of S relative to $\mathfrak{X}$.

Let F denote the boundary of S, K the minimal ideal of S and, for any subset A of S, let

$$P(A) = \{x \mid x \in S \text{ and } xA = A\}.$$

As is customary, AB denotes the set of all products ab with $a \in A$ and $b \in B$ and we generally write x in place of $\{x\}$. It will be convenient to abbreviate $P(S)$ by P. The structure of P is known in the following sense—supposing that $P \neq \square$ the set $P \cap E \neq \square$ and is indeed the set of left units of S, E being the set of idempotents. Moreover, if $e \in P \cap E$ then Pe is a maximal subgroup of S and the assignment $(x, y) \to xy$ is an iseomorphism (topological isomorphism) of $Pe \times (P \cap E)$ onto P. The following is a corollary to the principal result of [4]:

THEOREM 1. *If S is compact and convex and if $S \neq K$ then*

$$P(F) = P(S) \subset F.$$

It should be noticed that if S has a unit then $P = H_u$.

The quantifier *affine* will be applied to S if S is convex and if also $x(ty + (1-t)z) = txy + (1-t)xz$ and $(ty + (1-t)z)x = tyx + (1-t)zx$ for any x, y and $z \in S$ and any t with $0 \leqq t \leqq 1$. This differs a little from the definition in [1].

This is a particularly pleasant concept because of its generality and because of the host of examples of a simple geometric character. One such is the convex hull of the n roots of unity, using complex

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multiplication. It is indeed gratifying that such a familiar geometric form as a regular polygon should appear naturally in this context. But, not to slight modernity, the set of all $n \times n$ stochastic matrices is another example. A presently unpublished paper of M. J. P. Etter contains interesting results concerning such semigroups.

THEOREM 2. *If S is compact affine and if $P \neq \square$ then P is a closed extremal subset, $P \cap E$ is a closed convex extremal subset and for at least one $e \in P \cap E$ the set Pe consists entirely of extremal points.*

It follows from this that if S has a unit then H_u is a subset of the extremal points of S [1].

We recall that a subset T of S is *left simple*, if it is nonvoid and if $Tx = T$ for each $x \in T$. A result of Croisot states that each left simple subset is contained in a maximal such and that no two of these intersect. It is not hard to see that (S being compact) the maximal ones are closed and it may be observed that the set P is maximal right simple if it is not empty.

The next result is a kissing cousin of results of Kakutani, Klee and Peck (see the discussion in Chapter V of [2]) and extends a result in [1].

THEOREM 3. *If S is compact affine and if T is a left simple subset of S then*

$$A = \{x \mid x = xT\}$$

is a closed convex left ideal (and hence is nonvoid) while the set

$$\{y \mid x = xy \text{ for each } x \in A\}$$

is a closed convex subsemigroup.

Particularizing this it follows that if T contains the set of extremal points then S has a left zero, thus K consists entirely of left zeroes and is convex. Moreover, if S has a unit and if H_u contains the set of extremal points the S has a zero [1].

Now it is shown in [5] that if S is compact and convex (not necessarily affine) then $K \subset E$. An example in [1] shows that even if S is compact affine (with unit, then K need not be convex. This is a close shave since it follows from [6] (see Mathematical Reviews for an error) that K is iseomorphic with the cartesian product of two convex subsets of S.

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TULANE UNIVERSITY OF LOUISIANA