MAXIMUM THEOREMS FOR SOLUTIONS OF HIGHER ORDER ELLIPTIC EQUATIONS

BY SHMUEL AGMON

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The classical maximum modulus theorem for solutions of second order elliptic equations was recently extended by C. Miranda [4] to the case of real higher order elliptic equations in two variables. Previously Miranda [3] has derived a maximum theorem for solutions of the biharmonic equation in two variables. In the case of more variables it was observed by Agmon-Douglis-Nirenberg [2] that a maximum theorem holds in the special case of elliptic operators with constant coefficients with no lower order terms when the domain of definition is a half-space.

In this note we describe a very general maximum theorem for solutions of (complex) higher order elliptic equations in any number of variables. We shall obtain various estimates in the maximum norm which will contain as a special case the extension of Miranda's results to any number of variables.

We denote by G a bounded domain in E_n with boundary ∂G and closure $\overline{G}$. For a function $u \in C^j(\overline{G})$ we introduce the usual maximum norm:

$$\|u\|_j^{\overline{G}} = \max_{|\alpha| \leq j} \max_{x \in \overline{G}} | D^\alpha u(x) | . \tag{1}$$

Here $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multiple index of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and D^α is the corresponding partial derivative. Furthermore, for continuous functions u in $\overline{G}$ we introduce negative maximum norms $\|u\|_{-j}^{\overline{G}}$ $(j > 0)$ defined in the following manner. Write u in the form

$$u = \sum_{|\alpha| \leq j} D^\alpha f_\alpha \tag{2}$$

with $f_\alpha \in C^{|\alpha|}(\overline{G})$. Then:

$$\|u\|_{-j}^{G} = \operatorname{Inf} \max_{|\alpha| \leq j} \|f_\alpha\|_0^{\overline{G}}, \tag{3}$$

where the infimum is taken over all possible representations of the form (2).

Actually we are going to use negative norms for functions f defined on the (sufficiently smooth) boundary. If f has continuous derivatives up to the order $j \geqq 0$ on ∂G then one defines the jth maximum norm $\|f\|_j^{\partial G}$ in the usual way by means of local coordinates taking note of (1). Similarly it is obvious from (3) how one defines by means of local coordinates the negative norm $\|f\|_{-j}^{\partial G}$ ($j>0$ and f is continuous). Finally, for $u \in C^j(G)$ we also define the following L_p norm:

$$\|u\|_{j,L_p(G)} = \left(\int_G \sum_{|\alpha| \leq j} |D^\alpha u|^p dx \right)^{1/p}. \tag{4}$$

Now let

$$A = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha \tag{5}$$

be a (complex) elliptic operator of order $2m$ in $\overline{G}$. Suppose that $a_\alpha \in C^{|\alpha|}(\overline{G})$, G being of class C^{2m}. In the case of two variables we also assume that A satisfies the "roots condition" (see, for instance, [2]) a condition which is always satisfied for real elliptic operators or when the number of variables is at least three. We consider now functions $u \in C^{m-1}(\overline{G}) \cap C^{2m}(G)$ such that

$$\begin{aligned} Au &= 0 \qquad \text{in } G, \\ \frac{\partial^j u}{\partial n^j} &= \phi_j \qquad \text{on } \partial G,\ j = 0, \cdots, m-1, \end{aligned} \tag{6}$$

($\partial/\partial n$ denotes differentiation along the normal). The main result is the following

THEOREM I. *Let l be an integer such that $0 \leqq l \leqq m-1$. Then, for all functions u satisfying* (6) *the following estimate holds*:

$$\|u\|_l^{\overline{G}} \leqq c \sum_{j=0}^{m-1} \|\phi_j\|_{l-j}^{\partial G} + c_1 \|u\|_{L_1(G)}, \tag{7}$$

where c, c_1 are constants depending on A and G but not on u. If, moreover, the solution of the Dirichlet problem (6) *is unique in a suitable (small) class of functions then* (7) *holds with $c_1 = 0$.*

We note that the extension of Miranda's results corresponds to the case $l = m-1$. If $l < m-1$ then (7) contains negative norms on the right hand side (replacing these norms by the zero norm one obtains a weaker result). In particular, taking $l = 0$ and assuming uniqueness, one obtains the estimate:

$$\max_{\overline{G}} |u| \leqq c \sum_{j=0}^{m-1} \|\phi_j\|_{-j}^{\partial G} \leqq c_0 \sum_{j=0}^{m-1} \max_{\partial G} |\phi_j|. \tag{8}$$

Combining known existence results for the Dirichlet problem for smooth data with, for instance, the estimate (8), one obtains easily a solution of the Dirichlet problem (6) when the given data ϕ_j are merely continuous. It is an ordinary solution of the equation in the interior, continuous in $\overline{G}$, $u=\phi_0$ on ∂G, while the other Dirichlet data are taken in a generalized sense.

The method of proof of Theorem I uses an artifice introduced by Miranda in [4]. It consists in constructing a good "approximate solution" u_0 of (6) which takes the same Dirichlet data as u. For this purpose the Poisson kernels which resolve explicitly the Dirichlet problem for elliptic operators with constant coefficients in a half-space are used. These kernels were given in [2]. One then shows that the approximate solution u_0 satisfies (7) with $c_1=0$. Thus the problem is reduced to showing that the function $u_1=u-u_0$ (which has zero Dirichlet data) satisfies (7). This is done with the aid of the following L_p estimates for elliptic operators established recently by the author [1] (combined with Sobolev's inequalities).

THEOREM II. *Let* $u\in C^k(G)\cap L_q(G)$ *for some* $q>1$. *Let* $p>1$, $p'=p/(p-1)$. *Suppose that for all functions* $v\in C^{2m}(\overline{G})$ *such that* $\partial^j v/\partial n^j=0$ *on* ∂G $(0\leqq j\leqq m-1)$ *the following inequality holds*:

$$\left|\int_G u\overline{A}\bar{v}dx\right| \leqq C_u\|v\|_{2m-k,L_{p'}(G)}, \tag{9}$$

where C_u *is some constant depending only on* u. *Then*:

$$\|u\|_{k,L_p(G)} \leqq c_0C_u + c_1\|u\|_{L_1(G)} \tag{10}$$

where c_0, c_1 *are constants depending on the elliptic operator* A *and the domain but not on* u. *If, moreover, the solution of the Dirichlet problem* (6) *is unique then* (10) *holds with* $c_1=0$.

We shall illustrate the method of proof of Theorem I (in particular the manner in which Theorem II is used) in a special case where the construction of a good approximate solution u_0 is particularly simple. Consider a fourth order elliptic operator in the *plane* of the form:

$$A = \Delta^2 + A_1 \tag{11}$$

where A_1 is a lower order operator with variable coefficients. Take G to be a simply connected domain with sufficiently smooth boundary. Since by a conformal mapping the form of A remains unchanged (after division by some factor), we can assume without loss of generality that G is the unit-circle. As a suitable approximate solution one can choose here the solution u_0 of the biharmonic equation $\Delta^2u_0=0$

which takes the same Dirichlet data as u. This solution could be written down explicitly and it is easily verified by inspection that

$$\|u_0\|_l^{\bar{G}} \leqq K(\|\phi_0\|_l^{\partial G} + \|\phi_1\|_{l-1}^{\partial G}), \qquad l = 0, 1, \tag{12}$$

where K is some absolute constant. Put $u_1 = u - u_0$. We shall now use Theorem II to show that u_1 satisfies (7). Let A^* be the formal adjoint of A. By Green's formula it is readily seen that for all functions $v \in C^4(\bar{G})$ such that $v = 0$, $\partial v/\partial n = 0$ on ∂G:

$$\int_G u_1 \overline{A}^* \bar{v} dx = - \int_G A_1 u_0 \cdot \bar{v} dx.$$

Integrating the right hand side by parts and using Hölder's inequality we find readily for $l = 0, 1$ that

$$\left| \int_G u_1 \overline{A}^* \bar{v} dx \right| \leqq c_2 \|u_0\|_{l, L_p(G)} \|v\|_{3-l, L_{p'}(G)} \leqq c_3 \|u_0\|\|_l^{\bar{G}} \|v\|_{3-l, L_{p'}(G)}. \tag{13}$$

Applying Theorem II to u_1 (with $m = 2$, $k = l + 1$), we find that

$$\|u_1\|_{l+1, L_p(G)} \leqq c_4 \|u_0\|_l^{\bar{G}} + c_5 \|u_1\|_{L_1(G)}, \tag{14}$$

with $c_5 = 0$ if uniqueness holds. Choosing now $p > 2$ we have by Sobolev's inequalities:

$$\|u_1\|_l^{\bar{G}} \leqq K_1 \|u_1\|_{l+1, L_p(G)} \qquad (K_1 \text{ constant}). \tag{15}$$

Combining (15), (14) and (12) we get Theorem I in the special case considered.

References

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Hebrew University, Jerusalem, Israel and
Institute for Fluid Dynamics and Applied Mathematics,
University of Maryland