EXISTENCE IN THE LARGE OF TOTALLY GEODESIC SUBMANIFOLDS OF RIEMANNIAN SPACES

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1. Let M be a complete, C^∞ Riemannian manifold, x_0 a point of M, and N a subspace of M_{x_0}, the tangent space to M at x_0. In this note we give necessary and sufficient conditions that there be a *complete, immersed, C^∞, totally geodesic* submanifold of M through x_0, whose tangent space at x_0 is precisely N.

A once-broken geodesic starting from x_0 is said to be *admissible* if

(a) the tangent vector at any point of the geodesic belongs to N when parallel-translated back along the geodesic to x_0

(b) the second piece of the geodesic lies in a geodesic-convex neighborhood of the end-point of the first piece.

For $x \in M$, v_1 & $v_2 \in M_x$, let $R_x(v_1, v_2)$ be the skew-symmetric linear transformatiou of M_x induced by the Riemann curvature tensor. For each admissible, once-broken geodesic g let $N_g \subset M_{\text{end } g}$ be the subspace obtained by parallel-translating N along g to its end-point.

With these notations, we can state the result:

THEOREM. *Suppose that $R_{\text{end } g}(N_g, N_g)(N_g) \subset N_g$ for each admissible, once-broken geodesic g. Then, there is an immersed, C^∞, complete submanifold X of M whose tangent space at x_0 is N.*

The local version of this theorem was (more or less) given by Cartan [2]. Our job is to extend this globally by some sort of "analytic continuation." The proof, which is only sketched here, is inspired by the proof of Ambrose's isometry theorem [1]. One should notice then that conversely Ambrose's theorem is a consequence. Given two Riemannian manifolds M' and M'' that one hopes to show isometric, one looks for the graph of the isometry as a totally geodesic submanifold of $M = M' \times M''$.

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2. **The proof.** First, we are going to define X and a map $\phi: X \to M$ as a set of points, and then try to make X into a C^∞ manifold in such a way that ϕ is an immersion.

We need some local facts. Recall the definition of the map $\text{Exp}: M_x \to M$ for $x \in M$. If $v \in M_x$, $\text{Exp } v$ is the end point of the

geodesic starting at x, which points in the direction of v, and whose length is the length of v.

LEMMA 1.1. *For fixed $x \in M$, let U be an open set of M_x such that* (a) *for $v \in U$, $tv \in U$ for $0 \leq t \leq 1$ and* (b) Exp: $U \to M$ *is a diffeomorphism of U with an open neighborhood of x.*

Let N be a subspace of M_x. For $v \in N \cap U$, let $N_v \subset M_{\mathrm{Exp}\,v}$ be the subspace obtained by parallel translating N along the unique geodesic in Exp U *joining x to* Exp v. *Suppose that $R_{\mathrm{Exp}\,v}\,(N_v,\,N_v)(N_v) \subset N_v$.*

Then, Exp$(N \cap U)$ *is a totally geodesic submanifold of M. In particular, its tangent space at each* Exp v *is just N_v. For $v_1,\,v_2 \in N \cap U$ the result of parallel-translating N_v, along any path joining* Exp v_1 *to* Exp v_2 *in* Exp$(N \cap U)$ *is just N_{v2}. Any sufficiently small geodesic in* Exp(U) *whose end-points lie on* Exp$(N \cap U)$ *lies completely in* Exp$(N \cap U)$.

The proof is in [2].

Now, define X as follows: A point x of X is a pair (x, N_x) consisting of a point $x \in M$ such that $x = $ end-point of g and $N_x = N_g$ for at least one admissible, once-broken geodesic g. $\phi: X \to M$ is then defined by: $\phi(x) = x$.

For any sufficiently small open neighborhood U of 0 in N_x, define a subset $\boldsymbol{U}$ of X as follows:

$\boldsymbol{y} = (y, N_y) \in \boldsymbol{U}$ if and only if $y = $ Exp v for some $v \in U$, and N_y is the result of parallel-translating N_x along the geodesic $t \to $ Exp tv $(0 \leq t \leq 1)$, from x to y.

(Notice that the condition (b) for admissible geodesics and the local properties of totally geodesic submanifolds guarantee that Exp$(U) \subset \phi(X)$ if U is sufficiently small.)

Define the topology on X so that these sets $\boldsymbol{U}$ are a basis for the open sets. One proves, using the local Lemma 1.1, that

(a) the topology is Hausdorff;

(b) the intersection $\boldsymbol{U} \cap \boldsymbol{U}'$ of two such sets associated to (x, N_x) &$(x', N_{x'}) \in X$ correspond, under the identifications $\boldsymbol{U} \leftrightarrow U \subset N_x$, $\boldsymbol{U}' \leftrightarrow U' \subset N_{x'}$, to open sets of U and U'.

Then, we can use the identification of $\boldsymbol{U}$ with U, an open subset of Euclidean space, to make X into a C^∞ manifold. ϕ is then a C^∞ immersion of X in M, since $\phi: \boldsymbol{U} \to M$ can be identified with Exp: $U \to M$.

Now, Exp(U) is an imbedded, totally geodesic submanifold of M. Hence, if we make X into a Riemannian manifold in the obvious way, i.e. so $\boldsymbol{U}$ is isometric with Exp(U), then X is a totally geodesic sub-

manifold of M in the sense that ϕ maps geodesics of X into geodesics of M. It is complete, since all geodesics starting from $\mathbf{x}_0 = (x_0, N)$ can be indefinitely extended. The proof is then finished.

3. **Further remarks.** The theorem raises two general problems. First, what is the most general range of applicability in differential geometry of the method of "analytic continuation"? Second, how may global families of geodesics be put together to form nonsingular submanifolds?

If M and the metric are real analytic, the hypothesis of the theorem can be given a strictly local form. To do this, introduce $\Delta^n R_{x_0}$, the nth covariant derivative of R at x_0: For $v_1, \cdots, v_{n+2} \in M_{x_0}$, $\Delta^n R_{x_0}(v_1, v_2 : v_3, \cdots, v_{n+2})$ is a skew-symmetric endomorphism of M_{x_0}. The condition we have in mind is then that it always maps N into itself.

Following the pattern established by N. Hicks [3] in generalizing Ambrose's theorem, there seem to be no real difficulties in extending the constructions made above to the case of a complete affine connection, provided of course that one extends the class of admissible curves to include geodesics with an arbitrary number of components. It is not clear however whether one can prove that the resulting manifold is complete, since a powerful tool, the Hopf-Rinow theorem is not available.

BIBLIOGRAPHY

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