ON AN IDENTITY OF BLOCK AND MARSCHAK[1]

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Communicated by Edwin Moise, July 29, 1959

In the Bulletin of the American Mathematical Society[2] H. D. Block and Jacob Marschak proved for each choice of the positive integers m and n with $m \leqq n$ the identity

$$
\begin{aligned}
\sum_1 \{ (u_{s_1} + u_{s_2} + \cdots + u_{s_n})(u_{s_2} + u_{s_3} + \cdots + u_{s_n}) \cdots \\
(u_{s_{n-1}} + u_{s_n}) u_{s_n} \}^{-1} \\
= \{ (u_1 + u_2 + \cdots + u_m) u_2 u_3 \cdots u_n \}^{-1},
\end{aligned} \tag{1}
$$

where $u_1, \cdots, u_n$ denote indefinite numbers and where $\sum_1$ is extended over all the permutations $(s_1, s_2, \cdots, s_n)$ of $(1, 2, \cdots, n)$ which rank 1 before each of the numbers $2, 3, \cdots, m$.

In this paper I shall prove: *If p, q and n denote integers with $0 \leqq p \leqq q \leqq n$ and $n \geqq 1$, then*

$$
\begin{aligned}
\sum_2 \{ (u_{s_1} + u_{s_2} + \cdots + u_{s_n})(u_{s_2} + u_{s_3} + \cdots + u_{s_n}) \cdots \\
(u_{s_{n-1}} + u_{s_n}) u_{s_n} \}^{-1} \\
= \{ (u_1 + u_2 + \cdots + u_q)(u_2 + \cdots + u_q) \cdots \\
(u_p + \cdots + u_q) u_{p+1} \cdots u_n \}^{-1},
\end{aligned} \tag{2}
$$

where $\sum_2$ is extended over the permutations $(s_1, s_2, \cdots, s_n)$ of $(1, 2, \cdots, n)$ which rank 1 before 2; 2 before 3; $\cdots$; $p-1$ before p and finally p before each of the numbers $p+1, p+2, \cdots, q$.

The particular case $p=1$, $q=m$ yields (1).

In the proof of (2) I treat first the case $q=n$. Then $\sum_2$ is extended over the permutations $(s_1, \cdots, s_n)$ with $s_h = h$ $(1 \leqq h \leqq p)$, where $(s_{p+1}, \cdots, s_n)$ is an arbitrary permutation of $(p+1, \cdots, n)$. In this case we must show that

$$
\begin{aligned}
\sum_2 = \{ (u_1 + \cdots + u_n)(u_2 + \cdots + u_n) \cdots \\
(u_p + \cdots + u_n) u_{p+1} \cdots u_n \}^{-1}.
\end{aligned} \tag{3}
$$

In the case $p=n$ the sum $\sum_2$ consists of only one term namely

[1] This paper is sponsored by the United States Army under Contract No. DA-11-022-ORD-2059.

[2] H. D. Block and Jacob Marschak, *An identity in arithmetic*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 123–124. Without loss of generality we may choose $i=1$ in their identity and then this identity assumes the simpler form indicated in (1).

$(u_1+\cdots+u_n)^{-1}$. I may therefore assume that p is $\leqq n-1$ and that (3) has already been proved with p replaced by $p+1$. For each integer $h\geqq p+1$ and $\leqq n$ the contribution to $\sum_2$ of the permutations $(s_{p+1},\cdots,s_n)$ with $s_{p+1}=h$ is according to the induction hypothesis equal to

$$u_h\{(u_1+\cdots+u_n)(u_2+\cdots+u_n)\cdots (u_{p+1}+\cdots+u_n)u_{p+1}\cdots u_n\}^{-1},$$

so that

$$\sum_2=\{(u_1+\cdots+u_n)(u_2+\cdots+u_n)\cdots (u_{p+1}+\cdots+u_n)u_{p+1}\cdots u_n\}^{-1}\sum_{h=p+1}^{n}u_h,$$

which gives the required result (3).

Finally we treat the case $p\leqq q\leqq n-1$ and we may assume that (2) has already been proved with p replaced by $p+1$. We must prove that

$$(4)\qquad u_{p+1}u_{p+2}\cdots u_n\sum_2=[p+1,p+2,\cdots,q]/[1,2,\cdots,q],$$

where

$$[a_1,a_2,\cdots,a_t]=(u_{a_1}+u_{a_2}+\cdots+u_{a_t})(u_{a_2}+u_{a_3}+\cdots+u_{a_t})\cdots(u_{a_{t-1}}+u_{a_t})u_{a_t};$$

the right hand side means 1 if $t=0$.

By the induction hypothesis the contribution to $u_{p+1}\cdots u_n\sum_2$ of the permutations $(s_1,s_2,\cdots,s_n)$ which rank p before $q+1$ is equal to $[p+1,\cdots,q+1]/[1,\cdots,q+1]$; the contribution to $u_{p+1}\cdots u_n\sum_2$ of the permutations which rank $q+1$ between $h-1$ and h is for each integer h with $2\leqq h\leqq p$ equal to

$$u_{q+1}[p+1,\cdots,q]/[1,\cdots,h-1,q+1,h,\cdots,q]$$

and finally the contribution to $u_{p+1}\cdots u_n\sum_2$ of the permutations which rank $q+1$ before 1 is equal to

$$u_{q+1}[p+1,\cdots,q]/[q+1,1,\cdots q].$$

In this way we find

$$u_{p+1}u_{p+2}\cdots u_n\sum_2=\frac{[p+1,\cdots,q+1]}{[1,\cdots,q+1]}+u_{q+1}\sum_{h=1}^{p}\frac{[p+1,\cdots,q]}{[1,\cdots,h-1,q+1,h,\cdots,q]}.$$

It is therefore sufficient to prove that

$$
\tag{5} \frac{[p+1,\cdots,q+1]}{[1,\cdots,q+1]}+u_{q+1}\sum_{h=1}^{p}\frac{[p+1,\cdots,q]}{[1,\cdots,h-1,q+1,h,\cdots,q]} = \frac{[p+1,\cdots,q]}{[1,\cdots,q]}.
$$

This identity is obvious for $p=0$, so that I may assume that p is $\geqq 1$ and that (5) has already been proved with p replaced by $p-1$.

The term with $h=1$ occurring on the left hand side of (5) is equal to $(u_1+\cdots+u_{q+1})^{-1}$ times

$$
u_{q+1}\frac{[p+1,\cdots,q]}{(u_1+\cdots+u_q)[2,\cdots,q]},
$$

so that this term is a rational function of u_1 which possesses at $u_1=-(u_2+\cdots+u_{q+1})$ a simple pole with residue

$$
-[p+1,\cdots,q]/[2,\cdots,q]
$$

and at $u_1=-(u_2+\cdots+u_q)$ a simple pole with residue

$$
[p+1,\cdots,q]/[2,\cdots,q].
$$

The left hand side of (5) is therefore a rational function of u_1 which possesses at $u_1=-(u_2+\cdots+u_{q+1})$ a simple pole with residue

$$
\frac{[p+1,\cdots,q+1]}{[2,\cdots,q+1]}+u_{q+1}\sum_{h=2}^{p}\frac{[p+1,\cdots,q]}{[2,\cdots,h-1,q+1,h,\cdots,q]} - \frac{[p+1,\cdots,q]}{[2,\cdots,q]}.
$$

This expression assumes, if we replace $u_2, u_3, \cdots, u_{q+1}$ by $u_1, u_2, \cdots, u_q$, the form

$$
\frac{[p,\cdots,q]}{[1,\cdots,q]}+u_q\sum_{h=1}^{p-1}\frac{[p,\cdots,q-1]}{[1,\cdots,h-1,q,h,\cdots,q-1]} - \frac{[p,\cdots,q-1]}{[1,\cdots,q-1]}
$$

which is equal to zero according to formula (5) applied with p and q replaced by $p-1$ and $q-1$. Consequently the left hand side of (5) is a rational function of u_1 which has at $u_1=-(u_2+\cdots+u_{q+1})$ a simple pole with residue 0, so that this function is analytic at that point. This function has at $u_1 = -(u_2 + \cdots + u_q)$ a simple pole with residue $[p+1, \cdots, q]/[2, \cdots, q]$ and this is also the case with the function occurring on the right hand side of (5). All the terms occurring in (5) are analytic functions of u_1, apart of the points $u_1 = -(u_2 + \cdots + u_{q+1})$ and $u_1 = -(u_2 + \cdots + u_q)$, so that the difference between the two sides of (5) is a rational function of u_1 without poles which tends for $u_1 \to \infty$ to zero; this difference is therefore identically equal to zero. This completes the proof.

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