THE APRIL MEETING IN BERKELEY

The five hundred thirty-fifth meeting of the American Mathematical Society was held on April 20, 1957, at the University of California in Berkeley. Registrants numbered 144, including 124 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor A. L. Whiteman of the University of Southern California delivered the main address, on *Recent developments in the theory of cyclotomy*. He was introduced by Professor D. H. Lehmer.

Presiding over the sessions for contributed papers were Professors C. B. Allendoerfer, Leon Henkin, A. F. Moursund, and Bertram Yood. The abstracts of these papers are appended. Those having the abstract number followed by the letter "*t*" were read by title, the others in person. For papers having more than one author and presented in person, the letter (p) follows the name of the author who presented it. Dr. Cordes and Dr. Helmberg were introduced by Professor Klee.

ALGEBRA AND THEORY OF NUMBERS

548*t*. S. Chowla and E. G. Straus: *On the lower bound in the Cauchy-Davenport theorem.*

Let A, B be sets of residues (mod p) where p is prime. Let $A+B=\{a+b \mid a\in A, b\in B\}$ and let $n(S)$ denote the number of elements in S. The Cauchy-Davenport theorem states $n(A+B)\geqq \min\{p, n(A)+n(B)-1\}$. In the present note it is shown that if $n(A+B)<p-1$ then the lower bound $n(A)+n(B)-1$ is attained only if $n(A)=1$ or $n(B)=1$ or if A, B are in arithmetic progression with the same difference between consecutive terms. This theorem is applied to several number theoretic problems. For example, it is proved that if $(p-1, k)<(p-1)/2$ then every residue (mod p) can be expressed as the sum of no more than $[k/2]+1$ kth powers. (Received April 12, 1957.)

549. W. J. Coles: *On a theorem of van der Corput on uniform distribution.*

Van der Corput has shown (Acta Math. vol. 56 (1931) pp. 373–456) that the two-dimensional sequence $\{P_n\}$, $P_n=(\alpha_n, \beta_n)$, is uniformly distributed mod 1 if and only if for all integer pairs (u, v) other than $(0, 0)$ the one-dimensional sequence $(u\alpha_n+v\beta_n)$ is uniformly distributed mod 1. Let $F^{(n)}(x_0, x_1; y_0, y_1)$ be N^{-1} times the number of points of the type $(\alpha_n+a_n, \beta_n+b_n)$ $(n=1, \cdots, N)$, a_n, b_n integral, contained in the rectangle $x_0\leqq x\leqq x_1$; $y_0\leqq y\leqq y_1$. Let $D^{(N)}$ be the l.u.b. of $F^{(N)}$ over $0\leqq x_0<x_1\leqq 1$, $0\leqq y_0<y_1\leqq 1$. $D^{(N)}$ is the *discrepancy* of the set $P_1, \cdots, P_N$. Similarly one defines the discrepancy of one-dimensional sets. Let $D_{uv}^{(N)}$ be the discrepancy of the set of points $(u\alpha_n+v\beta_n)$ $(u=1, \cdots, N)$. *Theorem*: There is an absolute constant $\mathfrak{C}$ such that $D^{(N)}\leqq \epsilon+\mathfrak{C}(D_{10}^{(N)}+D_{01}^{(N)}+\sum_{(u,v)=1,u>0} f_{uv}(\epsilon)D_{uv}^{(N)})$ for any $\epsilon>0$, where $f_{uv}(\epsilon)$

$=\min(1/|uv|, 1/|\epsilon uv|^2)$. Since the series on the right converges uniformly in N (for fixed ϵ), van der Corput's sufficiency criterion follows from this quantitative criterion. The theorem is deduced from a similar theorem about an integrable function $r(x, y)$ periodic mod 1 in x, y. Also obtained is the following: there is an absolute constant $\mathcal{C}$ such that $|D| \leqq \mathcal{C}(\epsilon + \sum_{(u,v)=1, u>0} \sum_{1 \leqq uh \leqq 1/\epsilon, 1 \leqq vh \leqq 1/\epsilon} D_{uv}/|huv|)$ for any $\epsilon > 0$. (Received January 14, 1957.)

550. Lawrence Goldman and A. P. Hillman (p): *On statements of Poincaré and Schlesinger on linear differential equations.*

Let $\mathcal{K}$ be the field of rational functions of x with complex coefficients. Let $A(y)=0$ be a noncomposite homogeneous linear differential equation (h.l.d.e.) of order n with coefficients in a field $\mathfrak{F}$ normal over $\mathcal{K}$. Let $A_1(y)=0, \cdots, A_s(y)=0$ be the maximum set of distinct equations obtained by applying automorphisms of $\mathfrak{F}$ over $\mathcal{K}$ to $A(y)=0$. Poincaré (Acta Math. vol. 4 (1884) p. 205) stated without proof that every solution of $A(y)=0$ satisfies an h.l.d.e. $B(y)=0$, with coefficients in $\mathcal{K}$, of order ns. Schlesinger (*Handbuch der Linearen Differentialgleichungen*, vol. 1, pp. 217–219) gave a proof assuming that there cannot exist $y_1, \cdots, y_s$, not all zero, in an extension of $\mathfrak{F}$, such that $A_i(y_i)=0$ and $y_1 + \cdots + y_s = 0$. Although Poincaré's statement is correct, the following shows that such y_i may exist: Let $\alpha^3 + x\alpha^2 + (x-3)\alpha - 1 = 0$, $\mathfrak{F} = \mathcal{K}(\alpha)$, and $A(y) \equiv y' + y[\alpha'(2\alpha+1) - \alpha^2 - \alpha]/3\alpha(\alpha+1)$. One may choose $y_1 = [e^x/\alpha(\alpha+1)]^{1/3}$, $y_2 = \alpha y_1$, $y_3 = -(\alpha+1)y_1$. $B(y)$ exists of order 2. Sets of h.l.d.e. with dependent solutions are investigated in another paper. (Received February 18, 1957.)

551. C. V. Holmes: *Automorphisms of monomial groups.*

The automorphisms of monomial groups $\sum_n(H)$ consisting of finite substitutions were determined by Professor Oystein Ore, *Theory of monomial groups*, Trans. Amer. Math. Soc. vol. 51 (1942). If the restriction that the given set be finite be removed, monomial groups with infinite substitutions result. Denote such monomial groups by $\sum(H; B, C, D)$, where $d \leqq C$, $D \leqq B^+$, $d = \aleph_0$, B denoting the order of the given set which together with the group H generate the monomial group, C a cardinal such that all substitutions of the group have fewer than C factors different from the identity of H, D a cardinal such that each monomial substitution of the group maps fewer than D elements of the given set onto elements distinct from themselves, B^+ the successor of B. It is shown that any automorphism of the group $\sum(H; B, d, d)$ is the product of two automorphisms, an inner automorphism of the complete monomial group $\sum(H; B, B^+, B^+)$, and an automorphism T^+ which is generated by an automorphism T of H, T^+ maps any substitution into the same substitution with factors h_α replaced by $h_\alpha T$. A similar result is obtained for monomial groups $\sum(H; B, d, C)$, where $d < C < B^+$. In addition the automorphism groups of these monomial groups are determined. (Received February 20, 1957.)

552. C. T. Long: *Note on normal numbers.*

This note contains an easy proof of the fact that a real number α is normal to base r if and only if there exist positive integers $m_1 < m_2 < m_3 < \cdots$ such that α is simply normal to base r^{m_i} for each $i \geqq 1$. It is also proved that if $m_1, m_2, \cdots, m_s$ is an arbitrary collection of distinct positive integers, then there exists at least one real number simply normal to each of the bases $r^{m_1}, r^{m_2}, \cdots, r^{m_s}$ but not normal to base r. (Received February 20, 1957.)

553t. R. C. Lyndon: *On positive formulas in the predicate logic.*

Let L be the first-order predicate logic with both predicates and operation symbols, and possibly the predicate $=$, without special identity axioms. Consider L-formulas $A, B, C, \cdots$ in prenex normal form. A predicate ϕ is said to occur positively, or negatively, in A if some occurrence of ϕ in A is not, or is, preceded by the negation symbol. A is called positive in ϕ if ϕ does not occur negatively in A. Using Gentzen's extended Hauptsatz one obtains: *Theorem* 1. If $A \rightarrow B$ is L-provable, then, for some C, both $A \rightarrow C$ and $C \rightarrow B$ are L-provable, and no predicate occurs positively, or negatively, in C unless it so occurs in both A and B. This improves a result of W. Craig (Harvard thesis). Let ϕ_i, ϕ_i', for $i=1, 2, \cdots, p$, be n_i-place predicates, and $I(\phi_i, \phi_i')$ the formula stating that $\phi_i x_1 \cdots x_{n_i}$ implies $\phi_i' x_1 \cdots x_{n_i}$ for all $x_1, \cdots, x_{n_i}$. *Theorem* 2. If $A \wedge I(\phi_1, \phi_1') \wedge \cdots \wedge I(\phi_p, \phi_p') \rightarrow B$ is L-provable, where A does not contain the ϕ_i' nor B the ϕ_i, then, for some C, both $A \rightarrow C$ and $C \wedge I(\phi_1, \phi_1') \wedge \cdots \wedge I(\phi_p, \phi_p') \rightarrow B$ are L-provable, and C is positive in $\phi_1, \cdots, \phi_p$ and contains only predicates occurring in A. If, moreover, B results from A by replacing each ϕ_i by ϕ_i', then $A \leftrightarrow C$ is L-provable. (Received February 21, 1957.)

554. R. C. Lyndon: *On positive classes of relational systems and algebras.*

For terminology see Tarski, Indigationes Mathematicae 16, p. 572, and the preceding abstract. A relational system $\langle B, S_1, \cdots, S_n \rangle$ is called an enlargement of a system $\langle A, R_1, \cdots, R_n \rangle$ if $A = B, R_1 \subseteq S_1, \cdots, R_n \subseteq S_n$. Given a class K of (similar) relational systems, $\boldsymbol{E}(K)$ denotes the class of all enlargements of systems in K; $\boldsymbol{R}(K)$ denotes the class of all systems isomorphic to residue-class systems of systems in K; $\boldsymbol{A}(K)$ denotes the intersection of all arithmetical classes that include K. An arithmetical class K (in the wider sense) is called (i) relationally positive, (ii) equationally positive, or, simply, (iii) positive if it can be characterized by a set of L-formulas which are positive (i) in all predicates different from $=$, (ii) in $=$, or (iii) in all predicates. Theorem 2 of the preceding abstract implies: *Theorem* 1. For every arithmetical class K, $\boldsymbol{AE}(K)$ is relationally positive, $\boldsymbol{AR}(K)$ equationally positive, and $\boldsymbol{ARE}(K)$ positive. *Corollary* 2. An arithmetical class K is (i) relationally positive iff $\boldsymbol{E}(K) \subseteq K$, (ii) equationally positive iff $\boldsymbol{R}(K) \subseteq K$, (iii) positive if $\boldsymbol{E}(K) \cup \boldsymbol{R}(K) \subseteq K$. Applied to a class K of algebras, 2(ii) yields a result stated without proof by Łoś (*Mathematical interpretations of formal systems*, Amsterdam, 1955, p. 107). *Theorem* 3. There are arithmetical classes K such that $\boldsymbol{E}(K)$, $\boldsymbol{R}(K)$, $\boldsymbol{RE}(K)$ are not arithmetical. (Received February 21, 1957.)

555t. R. L. San Soucie: *Generalized weakly standard rings.*

If R is an arbitrary ring, not necessarily associative, define $K = [k \in R \mid (k, x, y) = (x, y, k) = 0$, all $x, y \in R]$. We call R *nuclear* if and only if $(x, y) \in K$, $(x, y, x) \in K$, all $x, y \in R$. If R is either commutative or associative, then R is nuclear. The main theorem of this paper is a partial converse: (*) A prime ring is nuclear if and only if it is either associative or commutative. Since a primitive ring is prime, and since a simple ring R with $R^2 \neq 0$ is also prime, we have: (**) Primitive rings and simple rings are nuclear if and only if they are either associative or commutative; a semi-simple nuclear ring is a subdirect sum of primitive nuclear rings. The class of nuclear rings includes the weakly standard rings of the author [Amer. J. Math. vol. 79 (1957) pp. 80–86], the accessible rings of Kleinfeld [Canadian J. Math. vol. 8 (1956) pp. 335–340], and the standard algebras of Albert [Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593]. Thus, Theorems (*) and (**) are generalizations of Albert's Theorem 14, Chapter V, and contain Kleinfeld's Theorems 1 and 3, as well as the author's Theorems 1–4, all as special cases. The proof of Theorem (*) does not depend upon the previous work, so the author is giving new proofs of these results in a more general framework. (Received February 6, 1957.)

ANALYSIS

556. W. G. Bade: *Multiplicity of spectral measures.*

Let T be a representation of the algebra of bounded Borel functions on a compact space in the algebra of bounded operators on a separable Banach space X. Recently Dieudonné (J. Math. Pures Appl. (1955) p. 155) has obtained a multiplicity theory for such representations. A new approach is obtained which achieved Dieudonné's results without the assumption of separability. This method also yields considerable new information about the invariant subspaces in X. (Received February 21, 1957.)

557*t*. W. G. Bade: *On unbounded finitely additive measures.*

The following theorem is proved: On every infinite field of sets there may be defined a real valued finitely additive measure whose range is unbounded. This settles in the negative a question raised by Hewitt (Mat. Tidsskr. B. (1951) p. 81) on whether every finitely additive measure whose domain is a σ-field is necessarily bounded. (Received February 21, 1957.)

558. A. V. Balakrishnan: *A generalization of the semigroups of stable distributions of P. Levy.*

Let $\phi_\alpha(it) = (-\cos \pi\alpha/2)|\cos \pi\alpha/2|^{-1}|t|^\alpha \exp -i\pi\alpha t/|t|$ for $-\infty < t < \infty$, $0<\alpha$, $\cos \pi\alpha/2 \neq 0$. Then it is shown that there is a one parameter semigroup of functions $S_\alpha(t)$, $0<t<\infty$, for each α, in $L_1(-\infty, \infty)$, such that $\phi_\alpha(it)$ is the logarithm of the Fourier transform of $S_\alpha(1)$. Moreover, for each $f \in L_1(-\infty, \infty)$, $\|S_\alpha(t) * f - f\| \to 0$ as $t \to 0$. For $0<\alpha \leqq 2$, the $S_\alpha(t)$ are the stable distributions of P. Levy; for $\alpha > 2$, they are no longer distributions. Again, if $T(\xi)$ is a strongly continuous one-parameter group of endomorphisms over a B-space X, with infinitesimal generator A, and $\|T(\xi)\| \leqq M$, then setting for each $x \in X$, $T_\alpha(t)x = \int_{-\infty}^{\infty} T(\xi)xS_\alpha(t, \xi)d\xi$, $0<t<\infty$, $T_\alpha(t)$ is a strongly continuous semigroup over X. For $x \in D(A^{n+1})$, $n = [\alpha]$, the infinitesimal generator C_α of $S_\alpha(t)$ can be represented $C_\alpha(x) = -((\cos \pi\alpha/2)/\Gamma(-\alpha))|\cos \pi\alpha/2|^{-1} \cdot \int_0^\infty (T(\xi)x - \sum_0^n (A^k x \xi^k/k!))\xi^{-\alpha-1}d\xi$. Among the additional properties proved is the spectral mapping theorem: $\sigma[C_\alpha] = \phi_\alpha(\sigma A)$. (Received February 18, 1957.)

559. F. H. Brownell: *Asymptotic distribution of eigenvalues for the lower part of the Schrödinger operator spectrum.* II.

Consider the eigenvalue problem $-\nabla^2 u(x) + V(x)u(x) = \lambda u(x)$ over $x \in R_n$ and $u \in L_2(R_n)$ with eigenvalues $\lambda_j \leqq \lambda_{j+1}$ repeated according to multiplicity starting with the least and with real eigenfunctions u_j. $N(\lambda) = \sum_{\lambda_j \leqq \lambda} 1$, the number of eigenvalues $\leqq \lambda$ satisfies $N(\lambda) \sim [(2\pi^{1/2})^n \Gamma(n/2+1)]^{-1} \int_{A(\lambda)} [\lambda - V(x)]^{n/2} d\mu_n(x)$, with $A(\lambda) = \{x \in R_n | V(x) \leqq \lambda\}$ and μ_n n-dimensional Lebesgue measure, as the right side $\to +\infty$ with increasing λ. This formula has been proved by Titchmarsh, Ray, and others under a variety of smoothness and growth conditions on $V(x)$, but all assuming $V(x) \to +\infty$ as $|x| \to +\infty$, which implies $\lambda_j \to +\infty$ as $j \to +\infty$. In an earlier abstract (Bull. Amer. Math. Soc. 62-4-566) we reported that this formula held under weaker conditions on $V(x)$ not given in detail. To clarify the record we specify that if $V(x) \to h$

finite real as $|x| \to +\infty$ in such a way that the right side integral $\to +\infty$ as $\lambda \to \bar{h}$, then, under mild additional conditions on $V(x)$, the formula still holds as $\lambda \to \bar{h}$, despite the fact that every $\lambda \geqq h$ will be contained in the spectrum and generally the continuous spectrum. (Received February 18, 1957.)

560*t*. F. H. Brownell: *Improved error estimates for the asymptotic eigenvalue distribution of the membrane problem for polygonal boundaries.*

Consider D an open, bounded, connected subset of the euclidean plane R_2, let B the boundary of D be the disjoint union of $p+1$ polygons, D being simply connected if $p=0$. Consider the eigenvalue problem $-\nabla^2 u = \lambda u$ over D, u having continuous second partials there, and u satisfying the Dirichlet or Neumann boundary condition on B. Then $N(\lambda)$, the dimension of the eigenspace corresponding to eigenvalues $\leqq \lambda$, satisfies $N(\lambda) = (4\pi)^{-1}\mu_2(D)\lambda + O(\lambda^{1/2})$ as $\lambda \to +\infty$, where $\mu_2(D)$ is the two-dimensional Lebesgue measure of D. Previously only $O(\lambda^{1/2} \ln \lambda)$ in this relation was known for such D. Similar improvements appear clear for dimension $n > 2$, also. The result comes from applying a trivial modification of the Tauberian theorem of Ganelius (Kungl. Fys. Salls. Lund Forhand. vol. 24, no. 20 (1954)) to the evaluation $\int_0^\infty e^{-t\lambda} dN_\nu(\lambda) = (4\pi)^{-1}\mu_2(D)t^{-1} - (-1)^\nu (8\pi^{1/2})^{-1} l(B) t^{-1/2} + [(1-P)/6 + \sum_k E_\nu(\alpha_k)] + O(\exp(-\delta^2/t))$ as $t \to 0^+$, where $\nu = 0$ or 1 designates the Dirichlet or Neumann boundary conditions respectively, $\delta > 0$, and $E_\nu(\alpha_k)$ depends only on the jump angle α_k of the tangent vector to B at the kth corner. This evaluation is obtained in exact analogue to the evaluation of $\int_\rho^\infty (1/\lambda(\lambda+\omega^2)) dN_\nu(\lambda)$ as $\omega \to +\infty$ found previously by the author (Journal Math. & Mech. vol. 6 (1957) pp. 119–166, particularly p. 152.) (Received March 4, 1957.)

561. J. B. Butler, Jr.: *A proof of Rellich's theorem for normal operators.*

Let $A(z)$ be a bounded operator on $\mathfrak{H}$, analytic in the complex parameter z at $z=0$. Wolf proved: If (i) μ_0 is an isolated eiqenvalue of $A(O)$ of multiplicity m, and (ii) $A(z_n)$ is normal on a sequence of points z_n with $\lim_{n\to\infty} z_n = O$, then there exists δ and an open set Δ containing μ_0 such that, for $|z| < \delta$, the spectrum of $A(z)$ inside Δ consists of m analytic functions $\mu_i(z)$, $i = 1, \cdots, m$, and the projections $P_i(z)$ corresponding to the eigenvalues $\mu_i(z)$ are analytic [Math. Ann. vol. 124 (1952)]. A short proof is derived from a theorem of Kato [J. Math. Soc. Japan vol. 4 (1952)] by establishing (I) $\mu_i(z)$ has a branch point at $z=O$ implies $P_i(z)$ has a pole. Further results: (II) (ii) is equivalent to: $A(z)$ is normal on an analytic arc through $z=O$. (III) If $A(z)$ is a closed operator having domain $\mathfrak{D}$ common with $A^*(z)$ and independent of z if $A^{-1}(O)$ is compact, and if $A(z)f$ and $A^*(\bar{z})f$ are analytic for $f \in \mathfrak{D}$, then (ii) implies that the spectrum of $A(z)$ consists of a denumerable sequence of functions $\{\mu_i(z)\}$ defined and analytic on an analytic arc through $z=O$. (Received February 18, 1957.)

562. Paul Civin: *A maximum modulus property of maximal subalgebras.*

Let B be a regular Banach algebra with a unit and with space of maximal ideals $\mathfrak{M}(B)$. Let N be a subalgebra of B which is not a maximal ideal of B and whose Gelfand representation separates the points of $\mathfrak{M}(B)$. Let π denote the mapping which represents B as a collection of continuous functions on $\mathfrak{M}(B)$. Suppose $\pi(N)$ is not dense in $\pi(B)$ and that N is maximal with respect to this property. The space $\mathfrak{M}(B)$

can then be embedded topologically in the maximal ideal space $\mathfrak{M}(M)$ of N. As so embedded $\mathfrak{M}(B)$ is the Šilov boundary of $\mathfrak{M}(N)$ with respect to $\pi(N)$, i.e. $\mathfrak{M}(B)$ is the smallest closed set of $\mathfrak{M}(N)$ on which each function in $\pi(N)$ attains its maximum modulus. (Received December 17, 1956.)

563. H. O. Cordes: *On a theorem of Hurwitz-Radon and certain integral estimates for partial differential operators.*

According to a theorem of Hurwitz and Radon it is always possible to find quadratic forms $q_i(x, x) = \sum_{j,e=1} a^i_{je} x_j x_e$, $i=1, \cdots, 4$, such that $\sum_{i=1}^4 q_i^2(x, x) = (\sum_{i=1}^4 x_i^2)^2$ for all quadruples (x_1, x_2, x_3, x_4) and the analogous statement holds for some vector-spaces of higher dimensions. It will be shown that this property can be used to give strong estimates of the form $\int_{r\leq r_0}(\Delta u)^2 r^\alpha dx \leq c\int_{r\leq r_0}(Lu)^2 r^\alpha dx$ where Lu is an elliptic differential operator. These estimates can be used to obtain an existence theorem for the Dirichlet problem of certain quasilinear elliptic differential equations. (Received February 22, 1957.)

564. C. J. A. Halberg, Jr., and H. G. Tucker (p): *Conditions for the Toeplitz summability of all powers of a bounded real sequence.*

Let $A = (a_{ij})$ be a regular (Toeplitz) matrix method of summation and let $x^{(k)} = \{x_j^k\}$ denote the "kth power" of a bounded real sequence. The principal result of this paper is embodied in the following theorem. A necessary and sufficient condition that $Ax^{(k)}$ converge for all k is that there exist a function $F(t)$ which is the limit almost everywhere of a sequence of functions $F_n(t)$, $F_n(t)$ being the sum of all a_{nj} over those j for which $x_j \leq t$. An application of this theorem is given. The sufficiency of the condition is applied to give an extremely simple proof of the strong law of large numbers in the case of bounded, independent, identically distributed random variables. (Received February 4, 1957.)

565t. Erwin Kreyszig: *On a class of systems of partial differential equations.*

Each solution (regular at the origin) of a system (1) $U_{x_\alpha x_\alpha} + U_{y_\alpha y_\alpha} + \phi_\alpha(x_\alpha, y_\alpha) U \equiv u_{z_\alpha z_\alpha^*} + F_\alpha(z_\alpha, z_\alpha^*)u = 0$, $z_\alpha = x_\alpha + iy_\alpha$, $z_\alpha^* = x_\alpha - iy_\alpha$, $\alpha = 1, 2$, can be represented in the form (2) $u = \int_{-1}^1\int_{-1}^1 E_1(z_1, z_\alpha^*, t_1) \times E_2(z_2, z_2^*, t_2)[f_1\{z_1(1-t_1^2), z_2(1-t_2^2)\} + f_2\{z_1(1-t_1^2), z_2^*(1-t_2^2)\}](1-t_1^2)^{-1/2}(1-t_2^2)^{-1/2}dt_1dt_2$ where f_1, f_2 are analytic functions of 2 complex variables regular at the origin. Cf. Bergman and Schiffer (Studies, pres. to R. v. Mises, 1954). A system (1) is said to be of class $\mathfrak{E}$ if in the representation (2) of its solutions generating functions of the form (3) $\mathfrak{E}_\alpha = \exp(\sum_{\mu=0}^{m_\alpha} q_{\alpha\mu}(z_\alpha, z_\alpha^*)t_\alpha^\mu)$, $\alpha = 1, 2$, can be chosen. Necessary and sufficient conditions were obtained in order that (1) be of class $\mathfrak{E}$. If (4) $f_1 = z_1^\beta z_2^\gamma$, $f_2 = z_1^\sigma z_2^{*\tau}$, $\beta, \gamma, \sigma, \tau = 0, 1, \cdots$, the corresponding solution of a system (1) of class $\mathfrak{E}$ satisfies 4 ordinary linear differential equations in each of the 4 variables; the coefficients are rational functions of the functions $q_{\alpha\mu}$; the order is independent of β, γ, σ, τ and depends on F_α only. The infinite set of these solutions forms a base of approximation; that is, each solution of a system of class $\mathfrak{E}$, regular in a convex domain $B_{z_1z_2}$ can be approximated by a linear combination of these solutions, uniformly in each closed subdomain of $B_{z_1z_2}$. (Received February 28, 1957.)

566t. Erwin Kreyszig: *On solutions of systems of partial differential equations.*

In the previous abstract an infinite set of independent particular solutions of systems of class $\mathfrak{E}$ was obtained which satisfy ordinary differential equations. There exists another infinite set of solutions having the same property. These solutions correspond to the meromorphic associated functions $f_1(z_1, z_2) = (z_1 - a_k)^{-k}(z_2 - b_n)^{-n}$, $f_2(z_1, z_2^*) = (z_1 - c_p)^{-p}(z_2^* - d_q)^{-q}$, $k, n, p, q = 1, 2, \cdots$, where a_k, b_n, c_p, d_n are arbitrary constants, different from zero (cf. formula (2) of the previous abstract). Resulting theorems: I. Solutions of systems of class $\mathfrak{E}$ with rational associated functions f_1, f_2 satisfy ordinary linear differential equations in each of the 4 variables. II. If a system of class $\mathfrak{E}$ has rational coefficients the singularities of its solutions with rational associated functions lie on algebraic manifolds of the $z_1 z_1^* z_2 z_2^*$-space. In this way the theory of ordinary differential equations can be used for characterizing properties of solutions of those systems, e.g. with regard to the behavior of the solutions in the neighborhood of singularities and outside of the domain of validity of the representation (2) (cf. the previous abstract). The system $U_{x_\alpha x_\alpha} + U_{y_\alpha y_\alpha} + U = 0$, $\alpha = 1, 2$, is of class E. Choosing $E_\alpha = i(z_\alpha z_\alpha^*)^{1/2} t_\alpha$ and associated functions (4) (cf. the previous abstract) the corresponding solutions are products of Bessel functions which satisfy ordinary differential equations of the fourth order. (Received February 28, 1957.)

567. T. J. McMinn: *Linear measures of sets of the Cantor type.*

From the closed unit interval remove the concentric open interval of length λ, $0 < \lambda < 1$. From each of the resulting closed intervals of length $(1-\lambda)/2$ remove the concentric open interval of length $\lambda(1-\lambda)/2$, etc. Let $T_{\lambda r}$ be the Cartesian product of the resulting set (the Cantor set for $\lambda = 1/3$) with the uniform magnification of it by a factor $r > 0$. Letting L and G be Carathéodory and Gillespie linear measures respectively, Randolph (J. London Math. Soc., 1941) has shown that $L(T_{\lambda 1}) = G(T_{\lambda 1}) = 0$ or ∞ according to whether $\lambda > 1/2$ or $\lambda < 1/2$ and $3/5^{1/2} \leqq L(T_{1/2\, 1}) \leqq 2^{1/2} < 3(2/5)^{1/2} \leqq G(T_{1/2\, 1}) \leqq 2$. We obtain $L(T_{1/2\, r}) = (1+r^2)^{1/2}$ and $G(T_{1/2\, r}) = 1 + r$ for $1/2 \leqq r \leqq 2$. We also determine $L(T_{1/2\, r})$ and $G(T_{1/2\, r})$ for $2 < r$ as well as obtain certain other results concerning L and G linear measures of homogeneous linear transformations of $T_{1/2\, 1}$. (Received February 4, 1957.)

568. J. S. Maybee: *A family of hyperbolic equations and the Cauchy problem for the wave equation.*

Let $\Box$ represent the wave operator in n-dimensions and let τ^m represent the mth partial derivative with respect to t. The paper studies equations of the form $\tau^m \Box u = f(x, t)$. By introducing two complex parameters μ and ν a Riesz kernel $W(t, x, \mu, \nu)$ is constructed for this equation with the following properties, (i) $\tau^m \Box W(t, x, \mu, \nu) = W(t, x, \mu-1, \nu-1)$, (ii) $\tau^m W(t, x, \mu, \nu) = W(t, x, \mu-1, \nu)$. (iii) $\Box W(t, x, \mu, \nu) = W(t, x, \mu, \nu-1)$ and (iv) W vanishes on the forward light cone with vertex at the origin. A two parameter semi-group of operators $J^{\mu,\nu} f$ is then introduced using the kernel W. It is shown that $J^{0,0} f = f$ and that, if m is chosen sufficiently large, $J^{1,1} f$ is well defined. The operators $J^{\mu,\nu}$ correspond to the Riesz operators introduced by Gårding. The Cauchy problem for the wave equation (which corresponds to the case $m = 0$) with the Cauchy data given on the plane $t = 0$ is solved by using the operator $J^{\mu,\nu}$. The method reproduces the solution of the wave equation given by Volterra for the case $n = 3$. (Received February 18, 1957.)

569. Werner Meyer-Koenig (p) and Karl Zeller: *The Tauberian gap theorem for Taylor's method.*

The series (1) $\sum u_k$ is called summable (T_α), i.e. summable by Taylor's method of order α, if $v_m = (1-\alpha)^m \sum_{k=m}^{\infty} \binom{k}{m} \alpha^{k-m} u_k$ $(m = 0, 1, \cdots)$ and $\sum v_m$ exist, and is called regularly summable (T_α), if it is summable (T_α) and if $\sum u_k z^k$ is regular at the point

$z=\alpha$. Let $0<\alpha\leqq 1/2$ and (1) a gap series with $u_k=0$ for $k\neq k_l$ $(l=0, 1, \cdots)$, where the $k_l\geqq 0$ are integers with $k_{l+1}-k_l>\theta\,(k_l)^{1/2}$ for a positive θ. Then if (1) is regularly summable (T_α) it is convergent. The proof uses a functional analytic method recently given by the authors in Math. Zeit. vol. 66 (1956) pp. 203–224. (Received February 20, 1957.)

570. R. S. Phillips: *Dissipative operators and dissipative hyperbolic systems.*

A linear operator L with domain $\mathfrak{D}(L)$ in a hilbert space H is said to be dissipative if $(Ly, y)+(y, Ly)\leqq 0$, $y\in\mathfrak{D}(L)$, and to be maximal dissipative (m.d.) if it is not the proper restriction of any other dissipative operator. An operator L is the infinitesimal generator of a strongly continuous semi-group of contraction operators on H if and only if L is a m.d. operator with dense domain. Further if L is m.d. with dense domain then so is L^*. Let Δ be a domain in real E_m and let $y(\chi, t)$ be a function of (χ, t) with values in a complex E_k. Consider the initial value problem: $y_t=E^{-1}[\sum(A^iy)_i+By]$, $\chi\in\Delta$, $t>0$; $y(\chi, 0)=f(\chi)$. Here E, A^i, and B are matrix-valued functions of χ, E is positive definite, the A^i are hermetian, and $B+B^*+\sum A^i\leqq\Theta$ for each $\chi\in\Delta$. Defining a minimal operator L_0 and a maximal operator L_1 corresponding to the spatial part of the above differential operator, one obtains semi-group generators either as m.d. extensions of L_0 or as m.d. restrictions of L_1. The generators of solutions for which energy neither enters from the interior of Δ nor through the boundary are the m.d. operators L such that $L_0\subset L\subset L_1$. A characterization of such operators is obtained by means of certain "maximal negative" subspaces of boundary data. (Received January 31, 1957.)

571. E. B. Tolsted: *Nontangential limits of subharmonic functions.*

Let $u(r, \theta)$ be a subharmonic function in the unit circle $r<1$, and let $F(\rho, \phi)$, the generalized mass distribution function associated with $u(r, \theta)$, satisfy the condition $\iint(1-\rho)dF(\rho, \phi)<+\infty$, where the integral is Stieltjes-Radon and is extended over the unit circle $\rho<1$. Littlewood showed that such a subharmonic function has a radial limit at almost all points of the circumference of the unit circle. Priwaloff published a generalization of Littlewood's theorem, allowing nontangential approaches to the circumference, but Tamarkin discovered an error in Priwaloff's proof and Zygmund constructed a counter-example to his theorem. In this paper certain averages of the subharmonic function over small circles are shown to have non-tangential limits, and it is shown that the subharmonic function itself has a non-tangential limit at almost all points on the circumference *provided* that the condition on the mass distribution function is strengthened to $\iint(1-\rho)f(\rho, \phi)\log^+ f(\rho, \phi)d(\rho, \phi)<+\infty$, where $f(Q)$ is a certain non-negative, Lebesgue-integrable function, called the density function associated with $u(r, \theta)$. (Received January 23, 1957.)

Applied Mathematics

572. S. D. Conte: *An implicit finite difference approximation to vibration problems in two space variables.*

The basic equation applying to the vibrations of a plate can be written in the form (1) $\Delta\Delta w+w_{tt}=0$ where Δ is the biharmonic operator. This equation is approximated by the finite difference system (2) $2^{-1}\Delta_x^4[w^*_{i,j,n+1}+w_{i,j,n-1}]+2\Delta_x^2\Delta_y^2 w_{ijn}+\Delta_y^4 w_{ijn}+(w^*_{ij,n+1}-2w_{ijn}+w_{i,j,n-1})/\Delta\bar{t}^2=0$, (3) $2^{-1}\Delta_y^4[w_{i,j,n+1}+w_{i,j,n-1}]=\Delta_y^4 w_{ijn}-(w_{ij,n+1}-w^*_{i,j,n+1})/\Delta\bar{t}^2$ where for example, $w_{ijn}=w(i\Delta x, j\Delta y, n\Delta t)$, $\Delta_t^2 w_{ijn}=(w_{i,j,n+1}-2w_{ijn}+w_{i,j,n-1})/\Delta\bar{t}^2$, $\Delta_x^4 w_{ijn}=(w_{i+2,j,n}-4w_{i+1,j,n}+6w_{ijn}-4w_{i-1,j,n}+w_{i-2,j,n})/\Delta\bar{x}^4$. The stability and convergence of (2) and (3) is investigated under appropriate boundary and initial conditions. (Received February 15, 1957.)

573. D. H. Hyers (p) and J. A. Ferling: *On the local uniqueness problem for water waves of permanent type*, I.

The question of the existence of periodic surface waves of permanent type in a channel of infinite depth was settled independently by T. Levi-Cività and N. Nekrasov around 1922–1924, for the case of small, but finite, amplitudes. However, little mention was made at that time of the *uniqueness* problem. The purpose of the present article is to initiate a study of the latter, first in the same case of small finite amplitudes. After the initial set-up of the problem, the present methods differ considerably in details from those used in the classical work mentioned above. They spring from some ideas of Lichtenstein, Liapounoff and others, as developed in Friedrich's Lectures on Functional Analysis (N.Y.U., 1950) in connection with bifurcation theory. In addition to the local uniqueness theorem, we also prove, as a sort of by-product, that a conjecture of Levi-Civita (Math. Ann. vol. 93 (1925) p. 284) is true at least locally, where the latter term is defined in a somewhat special sense. (Received February 28, 1957.)

574. H. M. Lieberstein: *On the generalized radiation problem of A. Weinstein.*

The following uniqueness theorem has been proved. Let $u^{(k)}$ be a solution of the two dimensional Euler-Poisson-Darboux (EPD) equation, $u^{(k)}_{xx}=u^{(k)}_{yy}+(k/y)u^{(k)}_y$, such that $u^{(k)}(x, 0)=f(x)$ and $u^{(k)}(x, x)=0, f(x)$ given in the interval $[0, a]$ and consider the triangle T with vertices (0, 0), (0, a), (a/2, a/2). Then there exists at most one $u^{(k)}(x, y)$, $-\infty<k<1$, having two continuous derivatives on a triangle G whose interior contains T and its sides except for the base line. Solutions of this boundary value problem with certain conditions on $f(x)$ have been given by A. Weinstein [Summa Brasiliensis Mathematical, vol. 3, Fasc. 7, September, 1955]. In another paper the author will construct solutions of a more general mixed problem for which the present uniqueness theorem applies. General solutions of the EPD equation as given by E. K. Blum [Duke Math. J., vol. 21 (1954) pp. 257–274] and Darboux [*Lecons sûr la thèorie gènèrale des surfaces*, Book IV, vol. 11, Paris, 1914–1915], together with an inversion of a certain integral equation yield the uniqueness for $-2<k<1$. It is shown that the uniqueness for all negative k follows through an induction using the Weinstein recursion, $u^{(k)}_y=yu^{(k+2)}$. Certain properties of the general solutions for negative k which are established by using propagation on the characteristic $y=x$ play an important role. Uniqueness for the classical radiation problem [Courant and Hilbert, Mathematische Physik, vol. II], which requires a singular solution of the EPD equation, is included as a special case. (Received February 18, 1957.)

Geometry

575. Albert Nijenhuis: *Differential-geometric operations on rings.*

From a commutative algebra R with unit over a field K the R-module Φ_p ($p=1, 2, \cdots$) is defined as the free module with generators $(r, s_1, \cdots, s_p)$; $r, s_1, \cdots, s_p\in R$, modulo relations: (a) "linearity" in each of the $p+1$ entries; (b) "triviality" of elements of K: $(r, s_1, \cdots, s_i, \cdots, s_p)$ for $s_i\in K$; (c) "product rule": $(r, s_1, \cdots, s_it_i, \cdots, s_p)-(rs_i, \cdots, t_i, \cdots, s_p)-(rt_i, \cdots, s_i, \cdots, s_p)$; (d) "skew-symmetry": $(r, s_1, \cdots, s_i, \cdots, s_j, \cdots, s_p)+(r, s_1, \cdots, s_j, \cdots, s_i, \cdots, s_p)$.

Multiplication by $r(s, t_1, \cdots, t_p) \to (rs, t_1, \cdots, t_p)$. There is a mapping $\wedge: \Phi_p \times \Phi_q \to \Phi_{p+q}$ by $(r, s_1, \cdots, s_p) \wedge (t, u_1, \cdots, u_q) \to (rt, s_1, \cdots, s_p, u_1, \cdots, u_q)$, and a mapping $d: \Phi_p \to \Phi_{p+1}$ by $d(r, s_1, \cdots, s_p) \to (1, r, s_1, \cdots, s_p)$; satisfying $d^2=0$. The element of Φ_p represented by $(r, s_1, \cdots, s_p)$ is thus $rds_1 \wedge \cdots \wedge ds_p$.—$R$ generalizes the C^∞ functions on a manifold; K the constant functions; Φ_p the p-forms.—The existence of the de Rham cohomology groups for R is now obvious. The derivations of R (relative to K) are in (1, 1) correspondence with the elements of Φ_1^*; vector p-forms are elements of $\mathrm{Hom}_R(\Phi_1, \Phi_p)$; cf. [A. Frölicher-A. Nijenhuis-Indag. Math. vol. 18 (1956) pp. 338–359]; and the theory of derivations on $\Phi = \bigoplus_{p=0}^{\infty} \Phi_p$ ($\Phi_0 = R$) and the existence of the bracket operator $[L, M]$ for vector forms are established as in the paper quoted. The de Rham groups, and concepts like almost product rings and almost complex rings need further study. (Received February 18, 1957.)

576. C. E. Watts: *On Riemannian connections.*

Let X be a C^∞-manifold, $C^\infty(X)$ the ring of real-valued C^∞ functions on X. Let T_q^p be the $C^\infty(X)$-module of cross-sections of the vector-bundle over X whose fiber at x is $T_x^* \otimes \cdots \otimes T_x^* \otimes T_x \otimes \cdots \otimes T_x$, where T_x is the tangent space at x, T_x^* its dual, and where T_x^*, T_x appear as factors p times and q times respectively. A Riemannian metric tensorfield induces symmetric $C^\infty(X)$-module isomorphisms $\gamma: T_1^p \to T_0^{p+1}$. An affine connection is considered as a $C^\infty(X)$-derivation $\delta: T_1^0 \to T_1^1$, and induces a derivation $D: T_0^1 \to T_0^2$. Let A^p be the module of exterior p-forms on X, and $a: T^p \to A_0^p$ the antisymmetrization operator. Then δ is a Riemannian connection iff $\gamma\delta = D\gamma$ and $da = aD$. A coordinate-free existence and uniqueness proof for the Riemannian connection is obtained, and explicit formulas for it are derived, involving only γ and d, or alternatively only γ and the bracket operation in T_1^0. (Received February 20, 1957.)

Logic and Foundations

577t. S. Feferman and R. L. Vaught: *Model-theoretic and decidability theorems concerning generalized products.*

For notation see preceding abstract. (I) and (II) include previous work of Mostowski (J. Symbolic Logic vol. 17 (1952) pp. 1–32), Vaught (Proc. Int. Cong. Math. 1954 vol. 2, pp. 409–410), and Feferman (Bull. Amer. Math. Soc. Abstracts 61-2-342 and 61-2-343), and with their aid (1)–(4) and (6), below, have been obtained. (1) *The generalized product and, hence, arbitrary cardinal and ordinal products and sums* (with $\mathfrak{J}$ fixed) *preserve* arithmetical equivalence (cf. Feferman, loc. cit.; for infinite ordinal sums this was found by Fraïssé (Zeit. f. math. Logik u. Grund. der Math. vol. 2 (1956) pp. 76–92)). (2) *If a sentence σ holds in the cardinal sum of $\langle \mathfrak{A}_i \mid i \in I \rangle$, then for some finite $J \subset I$, σ holds in the cardinal sum of $\langle \mathfrak{A}_i \mid i \in J' \rangle$ provided $J \subset J' \subseteq I$.* (3) *The decision problem for (the theory of) the generalized power [generalized weak power] of a system $\mathfrak{A}'$ to the exponent $\mathfrak{J}$ [$\mathfrak{J}'$] reduces to the decision problems for $\mathfrak{A}'$ and $\mathfrak{J}$ [$\mathfrak{A}'$ and $\mathfrak{J}'' = \langle P''(I), \subseteq, S_0', \cdots, S_{n-1}' \rangle$].* Consider any ordinal $\xi = \{\alpha \mid \alpha < \xi\} \cdot X <' Y$ means for some α, β, $X = \{\alpha\}$, $Y = \{\beta\}$, and $\alpha < \beta$; $+$ and $\cdot$ denote ordinal addition and multiplication. (4) *The decision problems for $\langle \omega^\xi, + \rangle$ and $\langle P''(\xi), \subseteq, <' \rangle$ are inter-reducible.* Write $X \sim Y$ if X and Y have the same power; $j \approx k$ if j and k have the same number of prime divisors. A reduction to decision methods of Presburger (also used for (4)) and Mostowski-Tarski for addition of cardinals yields: (5) *$\langle P''(I), \subseteq, \sim \rangle$ and $\langle P(I), \subseteq, \sim \rangle$ have decidable theories.* (6) *The theory of $\langle \omega, \cdot, \approx \rangle$ is decidable.* (Received February 15, 1957.)

578*t*. S. Feferman and R. L. Vaught: *Elimination of quantifiers in the theory of generalized products.*

$P(I)$, $P'(I)$, $P''(I)$ denote, respectively, the sets of all (i) subsets (of a set I), (ii) finite subsets or their complements, (iii) finite subsets. Given an indexed family $\langle \mathfrak{A}_i | i \in I \rangle$ of relational systems $\mathfrak{A}_i = \langle A^{(i)}, R_0^{(i)}, \cdots, R_{m-1}^{(i)} \rangle$ and finitary relations $S_0, \cdots, S_{n-1}$ among the members of $P(I)$, the *generalized product* $\mathfrak{B}$ of $\langle \mathfrak{A}_i | i \in I \rangle$ relative to $\mathfrak{J} = \langle P(I), \subseteq, S_0, \cdots, S_{n-1} \rangle$ is a system formed by the set B, consisting of all functions f on I such that $f(i) \in A_i$ for each $i \in I$, and by (infinitely many) relations $Q^{\theta}_{\phi_0, \cdots, \phi_{p-1}}$. Each $Q^{\theta}_{\phi_0, \cdots, \phi_{p-1}}$ is determined by a finite list $\phi_0, \cdots, \phi_{p-1}$ of formulas of the theory (=first order theory) of the $\mathfrak{A}_i$, having q free variables, and a formula θ of the theory of $\mathfrak{J}$ having p free variables. Members $f_0, \cdots, f_{q-1}$ of B are in the relation $Q^{\theta}_{\phi_0, \cdots, \phi_{p-1}}$, provided $U_0, \cdots, U_{p-1}$ satisfy θ in $\mathfrak{J}$, where $U_r = \{ i | i \in I$ and $f_0(i), \cdots, f_{q-1}(i)$ satisfy ϕ_r in $\mathfrak{A}_i \}$, $r = 0, \cdots, p-1$. Similarly, the *generalized weak power* $\mathfrak{B}' = \langle B', \cdots \rangle$ of a single system $\mathfrak{A}' = \langle A', e, R_0', \cdots, R_{m-1}' \rangle$ (having a distinguished element e) to the exponent $\mathfrak{J}' = \langle P'(I), \subseteq, P''(I), S_0', \cdots, S_{n-1}' \rangle$ is defined, where $S_0', \cdots, S_{n-1}'$ are finitary relations among the members of $P''(I)$, and B' is the set of all functions f on I to A' such that $\{ i | f(i) \neq e \}$ is finite. Two theorems, (I) and (II) are derived. (I) shows effectively that the set of relations of $\mathfrak{B}$ is closed under the Boolean and projective operations, (II) the same for $\mathfrak{B}'$. (Received February 15, 1957.)

Statistics and Probability

579. R. K. Getoor: *Additive functionals of a Markov process.*

Functionals of the form $\int_0^t v[x(\tau)]d\tau$ are considered where $x(t)$ is a temporally homogeneous Markov process in E^n and v is a non-negative measurable function on E^n. In investigating the distribution of such a functional one considers $r(t, x, y) = E\{\exp(-u\int_0^t v[x(\tau)]d\tau) | x(0) = x; x(t) = y\} p(t, x, y)$ where $p(t, x, y)$ is the transition density of $x(t)$ with respect to some Radon measure, m, which we assume exists. It is "intuitively clear" that if p satisfies a diffusion equation $\partial p/\partial t = \Omega p$ then r satisfies the equation $\partial r/\partial t = (\Omega - uv)r$. The purpose of this paper is to give an exact meaning to above statement under certain regularity assumptions on p. The first step in such a procedure is to give an unequivocable definition of the conditional expectation appearing in the definition of r. In accomplishing this we prove some theorems concerning conditional distributions of Markov processes which may be of some independent interest. (Received February 11, 1957.)

580. Herman Rubin: *Estimation of the location of a change of parameter.* Preliminary report.

Suppose for $t>0$ the random variable $X(a+t)-X(a)$ is a Poisson process with mean βt and $X(a)-X(a-t)$ is an independent Poisson process with mean γt, β, γ known, and $X(t)-X(u)$ observable for all t, u. This problem is treated as a translation parameter problem, and optimal invariant estimates obtained. This can be applied to the problem of estimating the location of a discontinuity in density (H. Chernoff and H. Rubin, *Proc. Third Berkeley Symposium in Probability and Statistics*). It is seen that, apart from a homogeneity factor, the distribution of the estimates depends only on β/γ, and hence the estimates are superefficient, as already noted in the above reference. Also the assumption that β, γ are known is unnecessary. (Research performed under OOR contract.) (Received February 20, 1957.)

Topology

581*t*. L. W. Anderson: *Complements in topological lattices.*

If L is a distributive lattice with 0 and 1 and if a and b are elements of L such that $a\vee b=1$ and $a\wedge b=0$ then the function $x\rightarrow(a\wedge x,\ a\vee x)$ is a lattice isomorphism of L onto $(a\wedge L)\times(a\vee L)$. Moreover, if L is a topological lattice then this function is a homeomorphism. *Corollary*: If L is a locally compact connected subset of the Euclidean plane and if L is a topological lattice with 0 and 1 and if a and b are elements of L distinct from 0 and 1 such that $a\vee b=1$ and $a\wedge b=0$ then L is topologically and lattice theoretically the same as the closed unit square endowed with the usual lattice operations. (Received February 11, 1957.)

582*t*. Anatole Beck: *On invariant sets.*

The solution of the following problem is announced: Given a one (real)-parameter group of homeomorphisms of the Euclidean plane onto itself, then the set of all points which are fixed under all these homeomorphisms is a closed set. It is wished to know precisely which sets can be obtained in this way. The answer is: All closed sets. It is found, in fact, that this result is true in a very broad collection of metric spaces, among which are all the metric real linear spaces. It is shown that in any metric space, every closed set (including the empty set) can be represented as the set of fixed points of some one-parameter group of homeomorphisms if and only if the empty set itself can be so represented and also that for some metric spaces (including the unit disc, the two-sphere, and the projective plane), the only closed set failing such a representation is exactly the empty set. (Received February 6, 1957.)

583. Gilbert Helmberg: *A theorem on equidistribution in compact groups.*

The paper gives a generalization of results contained in a paper by Beno Eckmann, *Über monothetische Gruppen*, Comm. Math. Helv. vol. 16 (1944). The following theorem is proved and discussed with respect to its consequences for the generation of a group and its subgroups by a set of elements: "Let G be a compact group with a countable base and $\{R^{(\lambda)}\}$ $(\lambda\in\omega)$ a complete system of inequivalent irreducible unitary representations of G, where $R^{(\lambda_0)}$ is the identity representation; furthermore let g_k $(k=1,2,\cdots,n)$ be n elements of G such that for each $\lambda\neq\lambda_0$ there is at least one element $g_{\bar{k}}$ for which the determinant $|R^{(\lambda)}(g_{\bar{k}})-E^{(\lambda)}|\neq 0$ ($E^{(\lambda)}$ = identity matrix in $R^{(\lambda)}$). Then the set $G'=\{g: g=g_1^{i_1}g_2^{i_2}\cdots g_n^{i_n},\ 0\leqq i_k<+\infty,\ k=1,2,\cdots,n)\}$ is equidistributed in G." (Received February 20, 1957.)

584*t*. L. E. Ward, Jr.: *Completeness in semi-lattices.*

A *semi-lattice* is a pair $(X,\ \leqq)$ where X is a set, $\leqq$ is a partial order on X and glb (x,y) exists for each $(x,y)\in X\times X$. The semi-lattice $(X,\ \leqq)$ is *complete* if glb A exists for each $A\subset X$. Lattice-theoretical results of Frink, Tarski, and Davis are generalized as follows: I. For the semi-lattice $(X,\ \leqq)$ to be complete, it is necessary and sufficient that, for each $x\in X$, the set $\{a: a\leqq x\}$ be compact in the interval topology. II. Let $(X,\ \leqq)$ be a semi-lattice, compact in its interval topology. If $f: X\rightarrow X$ is an isotone function, then the set P of fixed points of f is nonempty; further, $(P,\ \leqq)$ is a complete semi-lattice. III. For the semi-lattice $(X,\ \leqq)$ to be compact in its interval topology it is necessary and sufficient that every isotone $f: X\rightarrow X$ have a fixed point. (Received February 18, 1957.)

585. R. L. Wilder: *Montone mappings of manifolds.* II.

In an earlier paper [see Summary of Lectures and Seminars, Summer Inst. on Set Theoretic Topology, Madison, 1955, pp. 30–32] it was shown that an $(n-1)$-monotone image of an orientable n-gcm is (if nondegenerate and finite dimensional) an orientable n-gcm of the same homology type. It is the purpose of the present paper to extend this result, both to the orientable noncompact case and the locally orientable case, using proper mappings (i.e., such that counter-images of compact sets are compact). The orientable noncompact case does not offer great difficulty. For the locally orientable case the additional assumption must be made that for each point x of the image space, the counter-image of x lies in an orientable portion of the original space; the image-space is then a locally orientable n-gm of the same homology type as the original space. To show that the additional assumption is necessary, even in the compact case, an example is given of a classical 3-manifold S and a mapping f of S of the desired monotone type onto a space S' where S' is not even a generalized manifold. (Received February 19, 1957.)

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