THE APRIL MEETING IN NEW YORK

The five hundred thirty-third meeting of the American Mathematical Society was held on Thursday, Friday, and Saturday, April 4–6, 1957, at New York University. A Symposium on Orbit Theory (sponsored by the Society with the aid of the Office of Ordnance Research) was held on Thursday and Friday. About 550 persons attended, including 430 members of the Society.

The Symposium was divided into three sessions which met at 10:00 A.M. and 2:00 P.M. on Thursday, and at 10:00 A.M. on Friday. The first session was presided over by Professor J. B. Rosser. Participants in the Symposium were welcomed at this session by Colonel G. F. Leist, Commanding Officer of the Office of Ordnance Research of the U. S. Army. The following papers were presented: *Orbit stability in particle accelerators*, by Dr. E. D. Courant of the Brookhaven National Laboratory; *Motion of cosmic-ray particles in galactic magnetic fields*, by Professor Stanislaw Olbert of the Massachusetts Institute of Technology; *Störmer orbits*, by Dr. W. H. Bennett of the U. S. Naval Research Laboratory. Discussion was led by Professor Thomas Gold.

At the second session of the Symposium, the following papers were presented: *General theory of oblateness perturbations*, by Professor Paul Herget of the University of Cincinnati; *Fundamental problems in predicting positions of artificial earth satellites*, by Professor F. L. Whipple of Harvard University; *Cis-lunar orbits*, by Dr. K. A. Ehricke of the General Dynamics Corporation; *Satellite launching vehicle trajectories*, by Dr. J. W. Siry of the U. S. Naval Research Laboratory. The chairman for this session was Professor F. J. Murray, and the discussion was led by Dr. H. E. Newell.

The papers presented at the third session of the Symposium were: *Numerical determination of precise orbits*, by Professor W. J. Eckert of Columbia University; *Comments on general theories of planetary orbits*, by Professor Dirk Brouwer of Yale University; *Relativistic orbits in the central field of Birkhoff's theory of gravitation*, by Professor C. G. Fernândez of the University of Mexico. Professor R. E. Langer was the chairman for this session, and the discussion was led by Professor Jurgen Moser.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor Bernard Friedman of New York University addressed the Society at 2:00 P.M. on Friday on *Boundary conditions at infinity and virtual eigenvalues*, and Professor R. V.

Kadison of Columbia University delivered an address, entitled *A survey of the theory of operator algebras*, on Saturday at 2:00 P.M. These sessions were presided over by Professors K. O. Friedrichs and E. R. Kolchin respectively.

Sessions for contributed papers were held at 3:15 P.M. on Friday, and at 10:00 A.M. and 3:15 P.M. on Saturday. Chairmen at these sessions were Professors Melvin Henriksen, Fritz John, R. E. Johnson, P. D. Lax, Wilhelm Magnus, Louis Nirenberg, Mary E. Rudin, H. N. Shapiro, John van Heijenoort.

The Institute of Mathematical Sciences sponsored a tea on Friday afternoon.

Abstracts of the contributed papers follow. Those with "*t*" after the abstract number were presented by title. In the case of joint papers, the name of the author who read the paper is followed by (p). Miss Wood-Schot was introduced by Dr. R. M. Davis, Professor Preston by Professor A. H. Clifford, Mr. Miranker by Professor Lipman Bers, Dr. D. J. Newman by Professor M. S. Klamkin, Mr. Bzoch by Professor Pasquale Porcelli, Mr. Foote by Professor K. O. Friedrichs, Mr. Wendroff by Dr. G. M. Wing.

Algebra and Theory of Numbers

397. Shreeram Abhyankar: *Coverings of algebraic curves.*

One thesis of this paper is that the possibility of the "splitting of a single branch point by itself" is the basic fact which distinguishes the global ramification theory in the modular case from that in the classical case. Some of the consequences of this *possibility* are: Let the ground field k be algebraically closed of characteristic $p\neq 0$, then we have: (1) Given any one-dimensional algebraic function field K/k there exists x in K such that $x=\infty$ is the only valuation of $k(x)/k$ ramified in K. (2) There exist unsolvable unramified coverings of the affine line—which is a commutative group variety. (3) For nontamely ramified extensions, in no possible sense is the monodromie group generated by loops around the branch points. (2) is only a very special consequence of a general pattern which leads to the conjecture that (ramification theory in characteristic $p\neq 0$) = (ramification theory in characteristic zero for the *corresponding* situation) + (the class of quasi p-groups). Also the recent result of Lang-Serre to the effect that for a curve there are only a finite number of unramified coverings of a given degree is generalized to tamely ramified extensions with assigned branch points. (Received February 6, 1957.)

398*t*. E. H. Batho: *An analogue of Cohen's theorem for noncommutative local rings.*

Let R be a complete noncommutative local algebra over a field F such that R/J is finite dimensional and separable over F, where J is the Jacobson radical of R, (for terminology see Bull. Amer. Math. Soc. Abstract 61-6-633), then there is a related extension (S, ϕ) of R such that S is quasicyclic (Hochschild, Proc. Amer. Math. Soc. vol. 53 (1947) pp. 369–377). This generalizes a well known result of I. S. Cohen for complete commutative local rings. (Received February 10, 1957.)

399t. Harvey Cohn: *Some fundamental regions for which forms of dimension minus four are Eisenstein series.*

Using the variation of domain approach of the earlier work [Trans. Amer. Math. Soc. vol. 82 (1956) pp. 117–127], the author shows that the title is a valid property of a large class of regions with symmetries and generators essentially like those of the region for Hecke's modular functions ($zz'=a$, $z'=z+h$). (Received January 9, 1957.)

400t. H. S. M. Coxeter: *Groups defined by two relations.*

By an argument due to R. Frucht and B. H. Neumann (Publ. Math. Debrecen vol. 4 (1956) p. 191), the general metacyclic group of order mn can be presented in the form $S^m=T^n$, $T^{-1}S^uT=S^{u+1}$, $(1+u^{-1})^n\equiv 1 \pmod m$. Other interesting groups having two generators and two relations are: $T^n=1$, $T^{-1}ST=S^r$ ($r^n\neq 1$), of order $n\cdot|r^n-1|$; $S^m=T^n=(ST)^2$ ($k=2m+2n-mn>0$), of order $8mn/k$; $S^m=T^2=(ST)^n$ ($k>0$), of order $8mn(4n-k)/k^2$; $S^m=T^n$, $(ST)^2=1$ ($k>0$), of order $16mn(m+n)/k^2$; $S^m=(ST)^n$, $T^2=1$ ($k>0$), of order $16mn(m-n)/k^2$; $T^n=1$, $STS=TST$ ($n<6$), of order $24n^2/(6-n)^2$. (Received February 21, 1957.)

401. I. B. Fleischer: *Maximality and ultracompleteness in normed modules.*

The results are in the spirit of P. Conrad, *Embedding theorems for abelian groups with valuations,* Amer. J. Math. vol. 75 (1953) pp. 1–29; the methods similar to K. A. H. Gravett, *Valued linear spaces* Quart. J. of Math. vol. 6 (1955) pp. 309–315. The norm of the module M is non-archimedean and takes values in a linearly ordered set. The *residue class* module corresponding to an element λ of this set is the quotient module of elements with norm not exceeding λ by those with norm less than λ. An extension of M is *immediate* if it has the same value set and residue class modules; M is *maximal* if it admits no immediate extension. The existence and nonuniqueness of an immediate maximal extension is shown. M is *ultracomplete* if it is complete in the (not necessarily separated) topology defined by the inverse images of any collection of residue class modules. The equivalence of maximality and ultracompleteness is established in certain cases. For vector spaces, ultracompleteness implies the extensibility of bounded homomorphisms (cf. A. W. Ingleton, *The Hahn-Banach theorem for non-archimedean valued fields,* Proc. Cambridge Philos. Soc. vol. 48 (1952) pp. 41–45). This can be used to prove the uniqueness of the immediate maximal extension as well as to obtain some structure theory. Applications to linearly ordered vector spaces over linearly ordered fields are given. (Received February 20, 1957.)

402t. Lawrence Goldman and A. P. Hillman: *On sets of linear differential equations with dependent solutions.*

Let $\mathfrak{V}$ consist of all linear differential operators $D^n+a_1D^{n-1}+\cdots+a_{n-1}D+a_n$ with coefficients a_i in a differential field $\mathfrak{F}$ (of characteristic zero). $L_1,\cdots,L_r$ in $\mathfrak{V}$ are said to be dependent if there exist $y_1,\cdots,y_r$, not all zero, in an extension of $\mathfrak{F}$, such that $L_i(y_i)=0$ and $y_1+\cdots+y_r=0$. If the L_i are noncomposite, necessary and sufficient conditions for them to be dependent with no proper subset dependent is that they all have the same order n (i.e., degree in D) and that there exist $Q_1,\cdots,Q_r$ of order less than n in $\mathfrak{V}$ such that $Q_i(y_1)$ is a zero of $L_i(y)$ for every zero y_1 of $L_1(y)$ and $Q_1+\cdots+Q_r$ is the operator zero but no proper subset of the Q_i add up to zero. This result is obtained from Picard-Vessiot Theory. The following sufficient conditions for independence are also obtained: Let $A, B\in\mathfrak{V}$. Let $f_1,\cdots,f_r$ be distinct elements of $\mathfrak{F}$ and let $L_i = A + f_i B$ be noncomposite. If the f_i are constants and the order of A exceeds that of B or if $r=3$ and the order of A exceeds that of B by at least 2, then the L_i are independent. (Received February 18, 1957.)

403. Malcolm Goldman: *A coset-like decomposition relative to certain rings of operators.*

Let M and $N \subset M$ be factors of type II_1 (Murray and von Neumann, Ann. of Math vol. 37 (1936)) on a Hilbert space H such that the coupling constant (loc. cit.) of M is 1 and that of N is $1/n$, n an integer. Let f be a vector such that $\mathrm{Tr}\,(A) = (Af, f)$ for $A \in M$. We call such a vector a trace vector. Since $C=1$ for M it is known that there is a cyclic trace vector f for M. The author shows that there is a unitary $U \in M$ such that $U^n = 1$, the subspaces $[NU^k f]$ $k=1, \cdots, n$ are orthogonal for $k \neq j$ and they span H. He proves this by an inductive construction of trace vectors $f_1, \cdots, f_n$ for N such that $(Mf_i, f_j) = 0$ $i \neq j$. Letting $f_0 = \omega f_1 + \omega^2 f_2 + \cdots + \omega^n f_n$ where ω is a primitive nth root of unity he shows that $f_0/\|f_0\|$ is a trace vector for M and is cyclic under M. Therefore it is known that there is a unitary U with $Uf = f_0$. Thus if N has a complete orthonormal set of cyclic trace vectors on $[Nf]$ then M has such a family on H. Also if $UNU^{-1} \subset N$ then every $A \in M$ is a sum $A_1 U + A_2 U^2 + \cdots + A_n U^n$, $A_k \in N$. However, there are cases where no unitary $V \in M$ has the property $VNV^{-1} \subset N$. (Received Feburary 18, 1957.)

404*t*. Bruno Harris: *The structure of certain Jordan algebras.*

Let $\mathfrak{A}$ be the algebra of all $n \times n$ matrices over a field F, M an element of $\mathfrak{A}$ with invariant factors $\delta_1, \cdots, \delta_r$, δ_{i+1} dividing δ_i. The structure of $\mathfrak{A}(M)$, the algebra of matrices in $\mathfrak{A}$ commuting with M, is known (Jacobson, *Lectures in abstract algebra*, Vol. II): if $(F[x])_r$ is the algebra of $r \times r$ matrices with elements polynomials in an indeterminate x, $\mathfrak{C}$ the subalgebra of $(F[x])_r$ of matrices $(\alpha) = (\alpha_{ij}(x))$ such that $(\delta)^{-1}(\alpha)'(\delta)$ is in $(F[x])_r$ where $(\alpha)'$ is the transpose of (α) and $(\delta) = \mathrm{diag}\,(\delta_1, \cdots, \delta_r)$, and $\mathfrak{N}$ the ideal of $\mathfrak{C}$ of matrices of the form $(\alpha)(\delta)$, then $\mathfrak{A}(M)$ is isomorphic to $\mathfrak{C}/\mathfrak{N}$. We prove: 1. If $\mathfrak{J}$ is the Jordan algebra of $n \times n$ hermitian matrices over a field F of characteristic not two, N a nilpotent element with invariant factors $\delta_1, \cdots, \delta_r$, δ_{i+1} dividing δ_i, then $\mathfrak{J}(N)$, the Jordan algebra of elements of $\mathfrak{J}$ commuting with N, is isomorphic to $\mathfrak{S}/\mathfrak{K}$, $\mathfrak{S}$ the elements (α) of $(F[x])_r$ satisfying $(\delta)^{-1}(\alpha)^*(\delta) = (\alpha)$, $(\alpha)^*$ the conjugate transpose of (α), and $\mathfrak{K} = \mathfrak{S} \cap \mathfrak{N}$. 2. If $\mathfrak{J}$ is the Jordan algebra of $n \times n$ symplectic-symmetric matrices, N nilpotent with invariant factors δ_i, then $\mathfrak{J}(N) = \mathfrak{T}/\mathfrak{L}$, $\mathfrak{T}$ the matrices with $(\delta)^{-1}(\alpha)'(\delta) = (\alpha)$, $(\delta) = \delta_1(e_{12} - e_{21}) + \delta_3(e_{34} - e_{43}) + \cdots + \delta_{r-1}(e_{r-1,r} - e_{r,r-1})$, $\mathfrak{L}$ the matrices $(\alpha)(\delta)$. (Received February 20, 1957.)

405. Bruno Harris: *On a formula of Frobenius.*

If M is an $n \times n$ matrix over a field F with invariant factors $\delta_1, \cdots, \delta_r$ of degrees $d_1, d_2, \cdots, d_r$, then the algebra of $n \times n$ matrices commuting with M has dimension $\sum_{k=0}^{r-1} (2k+1) d_{k+1}$ (Frobenius; see Jacobson, *Lectures in abstract algebra*, vol. II). We generalize this formula to any central simple Jordan algebra $\mathfrak{J}$ over a field of characteristic not 2. First invariant factor degrees are defined for an element of such an algebra generalizing the usual definition and having the usual properties. The centralizer of the subalgebra generated by an element x is the subalgebra of all y in $\mathfrak{J}$ such that $R_y R_z = R_z R_y$ for all z which are polynomials in x, R_y denoting the multiplication by y in $\mathfrak{J}$: this is a generalization of "commutativity of two elements," due to Jacobson. $\mathfrak{J}$ is said to have degree n if on extending the base field to its algebraic closure, n is the maximum number of orthogonal idempotents in a decomposition of the identity. Theorem: If $\mathfrak{J}$ is central simple of degree n and dimension $n+n(n-1)s/2$, then the centralizer of the subalgebra generated by x has dimension $\sum_{k=0}^{r-1}(sk+1)d_{k+1}$. If $n\geqq 3$ then $s=1, 2, 4, 8$ and refers to the algebras of symmetric, complex hermitian, quaternionic hermitian, and octonion hermitian matrices, respectively. (Received February 20, 1957.)

406*t*. L. A. Hostinsky: *On splitting criteria.*

Within the framework of a complete modular lattice two types of criteria for uniform splitting (the existence of a complement) are developed. One deals with the concept of an almost periodic endomorphism defined as follows: The endomorphism η of $p/0$ is almost periodic on $x/0$ if $x\eta\leqq x\leqq p$ and if for every $y\leqq x$ there exists a positive integer $j(y)$ such that $y=y\eta^j$. It is shown that such an η is a uniformly splitting endomorphism of $p/0$ if and only if η induces a uniformly splitting endomorphism of p/x. The other criterion concerns the existence of a set of elements which are uniformly split by η. Application of this theory of splitting gives a further connection between homomorphisms and direct decompositions. Methods of proof depend largely upon properties of complements and decomposition endomorphisms. The investigation is based upon and extends certain previous results obtained by the author. (Received February 4, 1957.)

407. Abraham Karrass and Donald Solitar (p): *On free products of groups.*

Results are obtained for free products, among which are generalizations of theorems known for free groups. If a subgroup of a free product has finitely many conjugates H_i, then the H_i have a nontrivial intersection. In any group G, the intersection of a finitely generated subgroup with a subgroup of finite index is finitely generated. Let F denote a free product of finitely many groups; each of which is finite or more generally a finite extension of a free group of finite rank. F is hopfian. If the extensions are prime power corresponding to the same prime, then the lower central series of F intersects in the identity. A finitely generated subgroup containing a nontrivial normal subgroup of F must be of finite index in F. As a consequence we have the equivalence of the following for a subgroup H of F: (i) H is of finite index; (ii) H is finitely generated and has finitely many conjugates; (iii) H is finitely generated and contains the X^d-verbal subgroup of F for some $d>0$, i.e. the subgroup generated by the dth powers of elements of F. (Sponsored by National Science Foundation grant no. NSFG-2796). (Received January 30, 1957.)

408. M. D. Marcus: *On subdeterminants of doubly stochastic matrices.*

Let A be an n-square doubly stochastic (d.s.) matrix of rank k and let $1\leqq r\leqq k$. Let D_r be the sum of the squares of all r-square minors of A. *Then* $D_r\leqq\binom{k}{r}$ *and the equality holds for some r if and only if* $A=PBQ$ *where P and Q are permutation matrices and B is the direct sum of k* d.s. *matrices of rank* 1 (i.e. all of whose elements are the same). From this it follows that if A is as above then *A has* $\binom{k}{r}$ *r-square minors of value* ± 1 *if and only if* $k=n$ *and A is a permutation matrix.* (Received February 7, 1957.)

409*t*. Elliott Mendelson: *Some algebraic properties of nonstandard models for number theory.*

Let F be a first-order formal system for arithmetic, having as axioms all sentences which are true for the ring Z of integers. It is a well-known consequence of the Skolem-Löwenheim theorem that F has nonstandard models, that is, models which are not isomorphic with Z. Each nonstandard model R is an ordered integral domain having Z as a proper subring. Call such an integral domain R a *Peano ring*, and its quotient field K a *Peano field*. Peano rings have all "elementary" arithmetic properties in common with Z. The standard decomposition theorems for elements or ideals fail in R. K is the quotient field of no other Peano ring. Z is integrally closed in R, and R in K. Every element in $K-Q$ (where Q is the set of rationals) is transcendental over Q, and the absolute transcendence degree of K is infinite, but K is not a pure transcendental extension of Q. For any infinite cardinal m, there is a Peano field of absolute transcendence degree m. Given any transcendence base V of the reals over Q, there is a Peano field containing $Q(V)$ as an ordered subfield. (Received February 6, 1957.)

410. A. R. Amir-Moez (p) and Alfred Horn: *Singular values of a matrix.*

If A is any matrix with complex values, define the geometric singular values of A to be the non-negative square roots of the eigenvalues of A^*A. The real arithmetic singular values of A are the eigenvalues of $(A+A^*)/2$, and the imaginary singular values of A are the eigenvalues of $(A-A^*)/2i$. Relations between these singular values and the eigenvalues of A are discussed. In particular, a necessary and sufficient condition that there exist a matrix with real arithmetic singular values $\alpha_1 \geqq \cdots \geqq \alpha_n$ and eigenvalues λ_i such that $R\lambda_1 \geqq \cdots \geqq R\lambda_n$ is that $R\lambda_1 + \cdots + R\lambda_k \leqq \alpha_1 + \cdots + \alpha_k$ for $1 \leqq k \leqq n$, and $R\lambda_1 + \cdots + R\lambda_n = \alpha_1 + \cdots + \alpha_n$. A similar result holds for the imaginary singular values. If the geometric singular values of A are $\rho_1, \cdots, \rho_n$ and its real and imaginary arithmetic singular values are $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$, then $\rho_1^2 + \cdots + \rho_n^2 = \alpha_1^2 + \cdots + \alpha_n^2 + \beta_1^2 + \cdots + \beta_n^2$. (Received February 18, 1957.)

411. Morris Newman: *Some theorems about $p_r(n)$.*

A general congruence for the coefficients $p_r(n)$, defined by $\sum_{n=0}^{\infty} p_r(n)x^n = \prod_{n=1}^{\infty} (1-x^n)^r$ is derived, from which the Ramanujan congruences for partitions modulo 5, 7, 11 follow as a corollary. In addition it is shown that for $r=2, 4, 6, 8, 10, 14, 26$ the products $\prod_{n=1}^{\infty} (1-x^n)^r$ are lacunary, in the sense that arbitrarily long strings of zero coefficients occur, arbitrarily many times. (Received February 7, 1957.)

412. G. B. Preston: *The homomorphism theorems for partial algebras.*

A (λ, μ)-relation over a set A, where λ, μ are ordinals is a set $R=R(\lambda, \mu)$ consisting of ordered pairs of elements (a, b) where $a \in A^\lambda$ and $b \in A^\mu$, and such that if (a, b), $(a, b') \in R$ then $b=b'$. A partial algebra X is an ordered pair $X=(A, B)$ where each element of B is, for some λ, μ, a (λ, μ)-relation over A. Let $a=(a_\nu)$, $b=(b_\nu) \in A^\tau$; then we define $a+b$ to be $((a_\nu, b_\nu)) \in (A \times A)^\tau$. The set $Q \subset A^2$ is a congruence over X if Q is an equivalence over A and if, for (a, b), $(c, d) \in R = R(\lambda, \mu) \in B$, $a+c \in Q^\lambda$ implies $b+d \in Q^\mu$. Let $f: x \rightarrow xf$ be a single-valued mapping of A into M; if $a=(a_\nu) \in A^\tau$ denote by af the sequence $(a_\nu f) \in M^\tau$. By a homomorphism of $X=(A, B)$ into $Y=(M, N)$ we mean a pair (f, g) of single-valued mappings $f: A \rightarrow M$ and $g: B \rightarrow N$ such that if $(a, b) \in R \in B$ then $(af, bf) \in Rg$. By considering a noncommutative product of congruences over subalgebras of X we establish analogues of the isomorphism theorems and of Zassenhaus' Lemma for groups. In contrast to the situation in full algebras (in the sense of B. H. Neumann, Proc. London Math. Soc. (3) vol. 4 (1954) pp. 138–153) there are several natural analogues of these theorems, all of which remain true in an arbitrary partial algebra. We give conditions under which generalized Jordan-Hölder Theorems are valid. (Received February 18, 1957.)

413t. Irving Reiner: *A theorem on continued fractions.*

Let $R=K[x]$ be the ring of polynomials in an indeterminate x over a field K. For $f_1, \cdots, f_n \in R$, write $P/Q \sim [f_1, \cdots, f_n]$ to indicate that P and Q are the formal numerator and denominator, respectively, of the simple continued fraction $f_1+1/(f_2+ \cdots +1/f_n)$. Let $f \to f^*$ be any K-linear additive homomorphism of R into itself which is the identity on K. It is shown that if $f_1, \cdots, f_n \in R$ are such that P and Q lie in K, then also $[f_1^*, \cdots, f_n^*] \sim P/Q$. This implies that any identity of the form $[f_1, \cdots, f_n]=0$ depends only upon the additive structure of R, and not upon its multiplicative structure. (Received December 19, 1956.)

414. Fred Supnick: *Lattice points in n-spheres.*

Let G be the greatest integer such that *every* (closed, solid) n-sphere with diameter $n^{1/2}$ contains at least G lattice points (of an n-dimensional rectangular cartesian coordinate system). Let k denote the greatest integer less than or equal to $n/4$. In Warncke and Supnick, *On the covering of E_n by spheres*, (Proc. Amer. Math. Soc. vol. 8 (1957) pp. 299–303) the following result (a by-product of the proof of the main theorem) was obtained: $G \geqq 2^k$. In this paper a lower bound $H \equiv 1+\sum_{j=1}^{n-3}\sum_{i=1}^{k} A_{i,j}$ for G, $H \geqq [4^k(3n+7-12k)-(3n-2)]/9 \geqq 2^{2k-1}$ for $(n \geqq 4)$ is obtained, $(A_{i,j})$ being defined as follows: $A_{1,j}=1$ for all (positive integral) j; $A_{i,j}=0$ for each $j \leqq 4(i-1)$, whereas $A_{i,j}=\sum_{h=1}^{j-1} A_{i-1,h}$ for each $j>4(i-1)$. (Received February 18, 1957.)

415. E. J. Taft: *Invariant Wedderburn factors.*

Let C be a class of (finite-dimensional) algebras defined by identities. Let $A \in C$. Denote by R the radical of A. Let A/R be separable. Then the Wedderburn principal theorem for C states that A may be written $A=S+R$, where S is a separable subalgebra of A, $S \cong A/R$. Let G be a finite group each of whose elements is an automorphism or antiautomorphism of A. Assume that the characteristic of the base field is not a divisor of the order of the group G. Then ideal-theoretic conditions are given on C which guarantee the existence of Wedderburn factors S which are invariant under the operators of G. The results apply for C the class of associative algebras, alternative algebras, Jordan algebras, or Lie algebras (the last over a field of characteristic zero). The question of uniqueness is answered for the special case of self-adjoint (i.e., invariant under an involution) Wedderburn factors of an associative algebra over a field of characteristic zero. Any two such factors are conjugate via an orthogonal element, i.e., an element x that $xx^*=1=x^*x$. (Received January 25, 1957.)

416t. E. J. Taft: *The Wedderburn principal theorem for Jordan algebras.*

A proof is given of the Wedderburn principal theorem for Jordan algebras of characteristic not two, thus generalizing the result of Penico for characteristic zero (cf. *The Wedderburn Principal Theorem for Jordan Algebras*, Trans. Amer. Math. Soc. vol.

70 (1951). The main tools used are a structure theorem of Jacobson, and the existence of self-adjoint Wedderburn factors in alternative algebras (see previous abstract). Penico's proof consists mainly of a separate consideration of the five types of split algebras. Here his result is used for the case of split algebras of degree two. The other four cases (classes A, B, C, E) may be considered simultaneously. They are of the form $H(D_n)$, $n\times n$ hermitian matrices with entries from an alternative algebra D with an involution, $n\geqq 3$. Then self-adjoint Wedderburn factors of D give rise to Wedderburn factors of $H(D_n)$. A recent result of Jacobson shows that there are no new split algebras of characteristic $p\neq 2$. Hence Penico's proof, whose calculations involve only characteristic $\neq 2$, holds in this generality. The technique described here, however, sidetracks a considerable amount of calculation. (Received January 25, 1957.)

417*t*. R. L. Vaught: *Non-recursive-enumerability of the set of sentences true in all constructive models.*

Consider the first order logic without (or with) identity and with one (other), binary, predicate—at least. Let $\mathcal{K}_i$ $(i\leqq 4)$ be the class of all models $\langle A, R\rangle$ such that A and R are (0) arbitrary, (1) recursively enumerable (r.e.), (2) recursive, (3) primitive recursive, or (4) finite. Let V_i $(i\leqq 4)$ be the set of sentences true in all $\mathcal{K}_i$-models. Kreisel and Mostowski (cf. Fund. Math. vol. 42 (1955) p. 125) proved $V_0\neq V_1$ (assuming more complicated predicates). Trahtenbrot (Doklady Akad. Nauk SSSR vol. 90 (1950) p. 569; ibid. vol. 88 (1953) p. 953; Izv. Akad. Nauk SSSR Ser. Mat. vol. 20 (1956) p. 569) showed that V_4 is not r.e. (cf. also, Kalmar, Acta Math. Acad. Sc. Hung. vol. 2 (1951) p. 125), and, moreover, that if $V_0\subseteq X\subseteq V_4$, then X is not recursive. *Theorem* (*). *If* $V_1\subseteq X\subseteq V_4$, *then* X *is not r.e.* Thus, the title result holds, if "constructive" has any meaning such that the class $\mathcal{K}$ of all constructive models satisfies $\mathcal{K}_4\subseteq\mathcal{K}\subseteq\mathcal{K}_1$. Examples of sets X satisfying the hypothesis of (*) are V_1, V_2, and V_3, and, also, the sets W_i $(i=1, 2)$ of sentences valid in (1) all decidable theories, or (2) all finitely axiomatizable, complete theories. W_1 and W_2 were introduced by Szmielew-Tarski (Bull. Amer. Math. Soc. Abstract 55-11-577). That $V_1\subseteq W_i$ results from the lemma: *Every consistent, decidable theory has a* $\mathcal{K}_2$*-model.* (For terminology, cf. Tarski, *Undecidable theories*, Amsterdam, 1953.) (Received March 11, 1957.)

418. D. W. Wall: *Algebras with unique minimal faithful representations.*

In this paper an algebra means a finite dimensional algebra with unit over an algebraically closed field. The generalized uniserial algebras form a subclass of the QF-3 algebras which are those with unique minimal faithful representations. Thrall (Trans. Amer. Math. Soc. vol. 64 (1948) pp. 173–183) has shown that a QF-3 algebra can be defined as an algebra in which every primitive ideal is weakly subordinate to a sum of dominant ideals. Subclasses of the QF-3 algebras can be obtained by strengthening this definition by making one or more changes. Twelve subclasses of the QF-3 algebras are obtained in this manner and examples are given to show that these classes are distinct. Each of these classes contains the generalized uniserial algebras as a subclass. Necessary and sufficient conditions are given in order that an algebra in one class belong to a particular subclass. (Received February 21, 1957.)

419*t*. N. A. Wiegmann: *Some generalizations of Burnside's Theorem.*

Burnside's Theorem in the theory of group representations provides a necessary and sufficient condition that a semigroup of matrices over the complex field be irreducible; and this theorem has been generalized when the matrix elements lie in a division ring. When the elements are real quaternions, the following are among some results obtained: If $\mathfrak{A}$ is an irreducible semigroup of quaternion matrices, then any matrix M which commutes with every element of $\mathfrak{A}$ is of the form $P^{-1}(kI)P$ where k is complex and P is a nonsingular quaternion matrix. Let $\mathfrak{A}$ be a semigroup of quaternion matrices of degree n which is not similar to a complex set; if $\mathfrak{A}$ is irreducible, then $\mathfrak{A}$ has 1-rank n^2 and, conversely, if every semigroup similar to $\mathfrak{A}$ has 1-rank n^2, then $\mathfrak{A}$ is irreducible. (Received February 7, 1957.)

420t. K. G. Wolfson: *Ideals of the affine near-ring.*

Let $N(F, A)$ be the near ring of all affine transformations of the vector space A over the division ring F. Let S be the set of all constant transformations in $N(F, A)$. Denote the ring of all linear transformations of (F, A) by $T(F, A)$. For each ordinal $\nu \geqq 0$, $T_\nu(F, A)$ is the set of all transformations in $T(F, A)$ whose range has dimension less than $\aleph_\nu$, and $T_{-1}=0$. By an ideal is meant the kernel of a near-ring homomorphism. (Blackett, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 517–519.) It is shown (1) For each $\nu \geqq -1$, $T_\nu + S$ is an ideal in $N(F, A)$ and every ideal has this form. [$T_\nu + S$ is the set of elements $t+s$, t in T_ν, s in S.] (2) $N/T_\nu+S$ is isomorphic to T/T_ν. Hence, every proper homomorphic image of $N(F, A)$ is a dense ring of linear transformations. If (F, A) has finite dimension, S is the only proper ideal, so that $T(F, A)$ is the only proper homomorphic image of $N(F, A)$. (Received February 20, 1957.)

421t. K. G. Wolfson: *A generalization of H^*-algebras.*

Let $\{A_i\}$ be a set of complex Banach algebras. By the l_p direct sum of the $\{A_i\}$ is meant the set of all vectors $a=\{a_i\}$, $a_i \in A_i$ for which $\sum_i \|a_i\|^p < \infty$ with the coordinatewise operations and obvious norm. By a full matrix p-algebra of order n (n is a finite or infinite cardinal, $1 \leqq p \leqq \infty$) is meant the set of all $n \times n$ matrices $a=(a_{ij})$ with complex entries for which $\sum_{i,j} |a_{ij}|^p < \infty$. If $2 < p \leqq \infty$ then n can be finite only. Necessary and sufficient conditions are given that a Banach algebra be equivalent to an l_p direct sum of p-matrix algebras A_i (of order n_i) (same p in both places, n_i may be different in each summand). Use is made of the characterization of l_p spaces given by Bohnenblust (Duke Math. J. (1940) pp. 627–640). Such an algebra is dual (Kaplansky, Ann. of Math. (1948) pp. 689–701) if, and only if, all n_i are finite, or $p=2$. (The latter is the H^*-algebra of Ambrose, Trans. Amer. Math. Soc. (1945) pp. 364–386.) However, a sufficiently weakened remnant of duality may be used in the characterization. It is shown that the L_p algebras of a compact group have such representations (with all n_i finite). (Received February 20, 1957.)

ANALYSIS

422. J. W. Brace: *The weak completion of Banach spaces.*

A Cauchy net for the weak topology on a Banach space X converges to an element of X for the weak topology if and only if the net has a weak limit when embedded in an adjoint space of X under the natural embedding. (Received February 21, 1957.)

423t. R. C. Bzoch: *Concerning the existence and properties of the Lane integral.*

The object of this paper is to develop some of the properties of an integral defined recently by R. E. Lane (cf. Proc. Amer. Math. Soc. vol. 6, pp. 392–401). Our principal result is: Theorem A. If f is a bounded function on $[a, b]$ and g is of bounded variation on $[a, b]$, then a necessary and sufficient condition in order that the Lane integral $L\int_a^b f dg$ exist is that there exist a function f' on $[a, b]$ such that (i) the sigma-mean integral $\int_a^d f' dg$ exists, (ii) $f' = f$ on a subset G of $[a, b]$ such that $a \in G$ and $b \in G$, (iii) if $H = [a, b] - G$, then (f, g, H) is a singular graph, and (iv) if $x \in G$ and g is not continuous at x, then x is not exceptional point for f and g. An example is given to show that the theorem can not be extended to the case where f is unbounded. In the case of the necessity (iv) is redundant. The sufficiency part is similar to Lane's Theorem 3.3 (loc. cit.) but not equivalent to it. Also, theorems are developed on the structure of singular graphs and on the oscillatory properties of f on the set G. (Received January 3, 1957.)

424*t*. A. P. Calderón: *Uniqueness in the Cauchy problem for partial differential equations.*

Let $P(u) = 0$ be an mth order partial differential equation with highest order coefficients in $C_{1+\alpha}$, $\alpha > 0$, and remaining ones bounded and measurable. Assume that the characteristics of P are nonmultiple and let $u = u(x_1, \cdots, x_k) \in C_m$ be a solution of $P(u) = 0$ in a domain with vanishing Cauchy data on a noncharacteristic manifold M. Then if $k \neq 3$ or if $m = 2$, u vanishes identically in a neighborhood of M. Analogous result holds for systems provided $k = 2$ or $k > 4$. If the coefficients are assumed to be $m(r-1) + 2$ times continuously differentiable, where r is the number of equations in the system, then the result holds also for $k = 4$. The proof is based on a representation of a linear partial differential operator as a product of a fractional power of the Laplacian and a singular integral operator. The approximate functional calculus of singular integral operators allows then to reduce the equation to a form to which an inequality similar to the ones used by M. E. Heinz and N. Aronszajn can be applied. (Received February 5, 1957.)

425*t*. Y. W. Chen: *Reflection laws of differential equations of mixed type.*

Consider first the equation (*) $x^s u_{yy} + u_{xx} = 0$ with positive exponent s. The following theorem holds: Let $u(x, y)$ be a solution of (*) of class C_2 in a domain $0 < x < 1$, $a < y < b$, with continuous first and mixed second derivatives on $x = 0$. Let $u(0, y) = T_0(y)$ and $u_x(0, y) = T_1(y)$. Then, 1. T_0 and T_1 uniquely determines the solution $u(x, y)$; and 2. when one of the T_i, $i = 0, 1$, is zero, the other one is necessarily an analytic function of y, and $u(x, y)$ can be extended to an analytic function of two complex variables. Secondly the same equation is considered with negative $s > -1$. Under the additional assumption that u_{xyy} is continuous for $0 < x < 1$ and on $x = 0$, the above mentioned theorem is valid also in this case. The theorem generalizes the classical reflection principle of harmonic functions ($s = 0$). (Received February 27, 1957.)

426. Avron Douglis: *On linear, hyperbolic, partial differential equations of second order.*

Consider Cauchy's problem for a linear, hyperbolic, partial differential equation of second order in m independent variables. Its solution for any $m > 2$ is characterized by an integral equation of the form $u(P) = f(P) + {}_{C^P}\int K(P, Q) u(Q) dS_Q + {}_{D^P}\int J(P, Q) \cdot u(Q) dV_Q$, C^P being the part of the back characteristic conoid with vertex at P cut off by the initial surface and D^P its inside. The Cauchy data enter into the function $f(P)$. The kernels J and K are absolutely integrable over their domains, J vanishing for even m and K for odd m. The derivations of these formulas for even and for odd m run exactly parallel. (Received February 22, 1957.)

427t. Albert Edrei and W. H. J. Fuchs: *Deficient values which are also asymptotic.* II.

Continuing a previous investigation, the authors prove the following theorem. Let $f(z)$ be a meromorphic function of finite order λ, $1/2<\lambda<+\infty$, with $\delta(\infty)\geqq 1-\gamma$, $\sum_{a\neq\infty}\delta(a)\geqq 1-\gamma$. If γ is sufficiently small, there exists an integer p such that $|\lambda-p|<1/2$ and such that there are S $(1\leqq S\leqq p)$ finite deficient values which are also asymptotic. The deficiency of each of these values is at least a/p, where $a=a(\gamma, p)$ and $\lim_{\gamma\to 0}a(\gamma, p)=1$. The sum of the deficiencies of these S values is greater than a constant $b=b(\gamma, p)$ and $\lim_{\gamma\to 0}b(\gamma, p)=1$. (Received February 15, 1957.)

428. Charles Fox: *A composition theorem for general unitary transforms.*

The following theorem is proved: Let $k(a, x)$; $l(a, x)$ be a pair of kernels of a unitary transform (Chapter V, *Fourier transforms*, Bochner and Chandrasekharan) and let $p(a, x)$; $q(a, x)$ be another such pair of unitary kernels then $(\partial/\partial x)\int_0^\infty [k(x, y)]^- \cdot p(a, y)dy$ and $(\partial/\partial x)\int_0^\infty k(a, y)[p(x, y)]^- dy$ are also a pair of unitary kernels. (In these integrals $[f(x, y)]^-$ denotes the conjugate complex of $f(x, y)$.) It is also shown that these kernels can be related to $k(a, x)$ and $p(a, x)$ by means of the U and V operators introduced by Bochner (*Inversion formulae and unitary transforms*, Ann. of Math. vol. 55 (1934)). (Received February 7, 1957.)

429. F. W. Gehring: *The Fatou theorem for functions harmonic in a half space.*

It is supposed that $u(x, y, z)$ can be expressed as the difference of two functions each of which is both nonnegative and harmonic in $z>0$. Then $u(x, y, z) = kz +z/2\pi\iint_{-\infty}^{\infty}\{(x-\xi)^2+(y-R)^2+Z^2\}^{-3/2}d\mu(\xi, R)$ where μ is a measure defined over all Borel sets in the plane such that $\iint_{-\infty}^{\infty}(\xi^2+R^2+1)^{-3/2}|du(\xi, R)|<\infty$. Say that $D\mu(0, 0) =A$ if $1/\pi r^2\int_0^{2\pi}\int_0^r d\mu\to A$ as $r\to 0+$. Similarly we say that $D^*\mu(0, 0)=A$ if for all $\alpha<\beta, 1/\pi r^2\int_\alpha^\beta\int_0^r du\to((\beta-\alpha)/2\pi)A$ as $r\to 0+$ and if $|1/\pi r^2\int_\alpha^\beta\int_0^r du|\leqq M<\infty$ uniformly in α, β, for small r. The following results are analogues for theorems due to Fatou and Loomis. THEOREM 1. If $Du(0, 0)=A$, then $u(0, 0, z)\to A$ as $z\to 0+$. THEOREM 2. If $u(x, y, z)\geqq 0$ and if $u(0, 0, z)\to A$ as $z\to 0+$, then $Du(0, 0)=A$. THEOREM 3. If $D^*\mu(0, 0)=A$, then $u(x, y, z)\to A$ as $(x, y, z)\to(0, 0, 0)$ along each ray in $z>0$. THEOREM 4. If $u(x, y, z)\geqq 0$ and if $u(x, y, z)\to A$ as $(x, y, z)\to(0, 0, 0)$ along each ray in $s>0$, then $D^*\mu(0, 0)=A$. (Received February 21, 1957.)

430. Hugh Gordon: *Compatible topologies on Riesz spaces.* Preliminary report.

Let V be a Riesz space (i.e. a vector lattice). We call a topology on V compatible if: (1) it is compatible with the vector space structure of V; (2) the mapping $f\to f^*$ is continuous at 0; and (3) for each neighborhood U of 0 there is a neighborhood U' of 0 such that $f\in U'$, $g\in V$ and $0\leqq g\leqq f$ imply $g\in U$. These conditions imply the continuity of the lattice mappings $\vee$ and $\wedge$ of $V^*V\rightarrow V$. If V is a topological Riesz space (i.e. a Riesz space with a compatible topology), every continuous linear functional on V is relatively bounded, and the set V^* of all such functionals is a completely reticulated Riesz space. V^* is a topological Riesz space both under the topology defined by the set of all semi-norms $F\rightarrow|F|f$ (f positive in V) and under the strong topology. If M is a subset of the set V_R^* of all relatively bounded linear functionals on a Riesz space V, a necessary and sufficient condition for M to consist of all continuous linear functionals in a compatible topology is that: (1) M be a linear subspace of V_R^* and (2) $|F|\leqq|G|$, $F\in V_R^*$ and $G\in M$ imply $F\in M$. (Acknowledgment is made to the National Science Foundation under Contract NSF G 1981.) (Received February 19, 1957.)

431. L. J. Heider: *Measures on Boolean σ-algebras.*

Let $\mathfrak{A}$ be a Boolean σ-algebra. Let X_Δ denote Stone's zero-dimensional, compact Hausdorff space with element a of $\mathfrak{A}$ corresponding to open-closed subset Δ_a of X_Δ. Then $\mathfrak{A}$ as an σ-algebra is isomorphic to the algebra $\mathfrak{B}$ of Baire subsets of X_Δ, modulo the Baire sets of the first category. A measure ϕ on $\mathfrak{A}$ (and analogously for $\mathfrak{B}$) is a bounded real-valued function defined on $\mathfrak{A}$ with $\phi(a\vee b)=\phi(a)+\phi(b)$ for $a\wedge b=0$ in $\mathfrak{A}$. The usual distinctions of countably, finitely, and purely finitely additive measures are to be made. Then: the countably additive (Baire) measures on $\mathfrak{B}$ as restricted to the open-closed subsets Δ_a determine all measures on $\mathfrak{A}$. Each non-negative Baire measure ϕ on $\mathfrak{B}$ has a unique decomposition $\phi=\phi_C+\phi_P$ into a non-negative Baire measure ϕ_C identically zero on Baire sets of the first category, plus a Baire measure ϕ_P minimal on these sets in the sense that $0\leqq\psi\leqq\phi_P$ with ψ a Baire measure identically zero on the Baire sets of the first category implies that ψ is the zero measure. Under the correspondence $\Delta_a\leftrightarrow a$, this decomposition of Baire measures on $\mathfrak{B}$ determines a corresponding unique decomposition of measures on $\mathfrak{A}$ into sums of a countably additive measure and of a purely finitely additive measure on $\mathfrak{A}$. For any measure ϕ on $\mathfrak{A}$ this latter decomposition can be obtained directly and, indeed, either so as to obtain first the countably additive portion of the measure or so as to obtain first the purely finitely additive part of the measure. (Received February 13, 1957.)

432. W. P. Jurkat: *On relations between Borel's and Cesàro's methods of summation.*

Using the complex methods introduced in previous papers the following typical result is obtained: The two conditions (i) $f(z)=e^{-z}\sum_{n=0}^{\infty}s_n z^n/n!=o(1)$ as $z\rightarrow\infty$, $|z|-\mathrm{Re}\,z=O(1)$, (ii) $\sum_{k=0}^{n}s_{N+k}=o(N^{1/2})$ as $N\rightarrow\infty$, $n=O(N^{1/2})$ are equivalent for bounded sequences s_n. This theorem contains most of the Tauberian Theorems for Borel summability. Actually (ii) implies (i) without restriction, and the converse can be proved even under more general conditions as $\sum|s_k|^2=O(n^{1/2})$ summing from n to $n+n^{1/2}$. There are several generalizations and a discussion of the relationship to well known results of Hardy-Littlewood in this field. (Received February 18, 1957.)

433. M. S. Klamkin: *An application of the Gauss multiplication theorem.*

S. K. L. Rao has given a derivation of Legendre's duplication formula by means of Mellin transforms. On attempting to extend the method to a proof of the Gauss multiplication theorem, one is led to the evaluation of some interesting integrals involving modified Bessel functions. (Received February 18, 1957.)

434. Paul Koosis: *A completeness theorem.*

$m \neq 0$ is a measure with (finite) supporting interval I. Let $\{a_k\}$ be all the complex zeros of $\hat{m}(\lambda) = \int e^{i\lambda x} dmx$, a_k occurring with mult. n_k+1. $X = \{x^n \exp(-ia_k x) \mid 0 \leqq n \leqq n_k; a_k\}$. Theorem: X is unif. complete on any interval of length $<$ that of I. Proof: If not, there is a meas. $p \neq 0$ of supp. int. $J \subset I$ prop., with $F(\lambda) = \hat{p}(\lambda)/\hat{m}(\lambda)$ entire, $\therefore$ of order $\leqq 1$. W.l.o.g., $I = [0, A]$, $J = [\delta, A-\delta]$, $\delta > 0$. $e^{-i\delta\lambda}\hat{p}(\lambda)$ is bdd. in $y\lambda \geqq 0$. Prev. result of author (cf. §2.3 of paper in Comm. Pure Appl. Math. February, 1957) then shows that given $\epsilon > 0$ there exists $c > 0$ so that for $re^{i\beta} = \lambda - ic$ & $0 < \beta < \pi$, β bdd. away from 0 & π, $|e^{-i\delta\lambda}F(\lambda)|$ is $O(\exp[\epsilon r/\sin\beta])$. $\therefore$ $F(\lambda) \to 0$ as $\lambda \to \infty$ in a suff. small sector ab't pos. imag. axis. Sim. in a sector ab't neg. imag. axis. Phragmén-Lindelöf Th. shows then that $F(\lambda)$ is bdd. $\therefore$ $F(\lambda) = $const & const is zero. $\therefore$ $p \equiv 0$, contr. This clears up an an obscurity in a paper of L. Schwartz (cf. p. 157, l. 7–10 & ftnote, Annales Fac. Sci. Toulouse, VI, 1942). (Received February 19, 1957.)

435. Walter Koppelman: *The Riemann-Hilbert problem for finite Riemann surfaces.* Preliminary report.

Suppose that M is a Riemann surface, topologically equivalent to a sphere with h handles and m holes, whose boundary ∂M has a Hölder continuously turning tangent. Assume that Λ, γ are continuous functions on ∂M, where Λ is complex-valued, Hölder continuous, with $|\Lambda| = 1$ and where γ is real. The boundary value problem Re $(\Lambda w) = \gamma$ for analytic function multiples w of a given divisor d in M has a solution if, and only if $\int_{\partial M} \bar{\Lambda}\gamma dV = 0$ for every analytic differential multiple of $1/d$ which satisfies Re $(\bar{\Lambda}(dv/dz)(dz/dt)) = 0$ on ∂M. For $\gamma = 0$, there exists a Riemann-Roch theorem which relates the dimensions of the two linear spaces of the functions and differentials involved. This is a generalization of part of a result obtained by I. N. Vekua (Mat. Sbornik, N.S. vol. 31 (73) (1952) pp. 217–314). The proof is first given for analytic boundaries, where the double F of M is used to reduce the problem to a Dirichlet problem. The Riemann-Roch theorem above is a consequence of the Riemann-Roch theorem for F. For nonanalytic boundaries, new uniformizers are introduced so that the boundary curves are analytic relative to the new co-ordinates. (Received November 29, 1956.)

436. Saul Kravetz: *On the geometry of Teichmueller manifolds and the structure of the mapping-class group.* III. Preliminary report.

The author continues his investigation [Bull. Amer. Math. Soc. 59 (1953) p. 541] of the geometry of Teichmueller manifolds. The negative curvature, as defined by H. Busemann ([1] Acta Math. 80 (1948) pp. 259–310), of the Teichmueller manifolds is proved. It follows [1, Theorem 4.11] that every finite group of self-isometries of a Teichmueller manifold has a fixed-point. This solves *Teichmueller's Fixed-Point Problem*, i.e. Does every finite subgroup of the mapping-class group acting on a Teichmueller manifold have a fixed-point? Teichmeuller noted [Abh. der Preuss. Akad. der Wiss., Math.-Naturw. Klasse (1939) No. 22, §149] that this fixed-point problem is equivalent to the *Hurwitz-Nielsen Realization Problem*, i.e. Given a finite subgroup of the mapping-class group of a compact topological surface, does there exist a complex-analytic structure on the surface (making it a Riemann surface) admitting a group of conformal self-mappings whose natural image in the mapping-class group is just the given finite subgroup of the mapping-class group? (Received February 22, 1957.)

437. Klaus Krickeberg: *Distributions and Lebesgue area.*

Let G be open in R_n, and write points of R_n as (x, y) with $x \in R_m$, $y \in R_k$, $m+k=n$.

Given a subset B of G, a function f on G and y in R_k, denote by B_y the set $\{x: (x, y) \in B\}$ and by f_y the function $f_y(x) = f(x, y)$. Consider a linear differential operator D_0 over R_m with constant coefficients, and the operator D over R_n defined by $(Df)(x, y) = (D_0 f_y)(x)$ if f is infinitely differentiable. Assuming f to be locally summable in G, the distribution Df in the sense of L. Schwartz is a measure μ if and only if $D_0 f_y$ is a measure μ_y for almost all y, and the total variation $|\mu_y|(B_y)$ is a summable function of y for every bounded Borel set $B \subseteq G$. In this case $\mu(B) = \int_{R_k} \mu_y(B_y) d\lambda_k(y)$ where λ_k denotes the k-dimensional Lebesgue measure. It follows that a measurable function f on G is locally summable and its derivatives $\partial f/\partial x_i$ are measures ν_i if and only if f is locally of bounded variation in the sense of Tonelli and Cesari (TC). The Lebesgue area of the nonparametric hypersurface $x_{n+1} = f(x_1, \cdots, x_n)$ which by a generalization of a theorem of Cesari is finite if and only if $\lambda_n(G) < +\infty$ and f is of bounded variation TC, is then given by $\int_G ((d\lambda_n)^2 + (d\nu_1)^2 + \cdots + (d\nu_n)^2)^{-1/2}$, i.e. the total variation of the vectormeasure $(\lambda_n, \nu_1, \cdots, \nu_n)$ over G. (Received February 18, 1957.)

438. P. D. Lax: *An asymptotic Huygens' principle for nonlinear equations.* Preliminary report.

Let $u(x, t)$ be a solution of a first order quasilinear hyperbolic system $u_t + A(u)u_x = 0$. Suppose that initially u is zero outside of a finite interval. If the number of variables is two, the Riemann invariants are constant along the characteristics and therefore the solution is zero at all points which cannot be connected by at least one characteristic to the interval containing the initial disturbance. This is an example of Huygens' Principle. For systems with a larger number of dependent variables, such a Huygens' Principle has an asymptotic validity in the initial amplitude. In the presence of shocks the result is valid up to terms of third order. (Received February 11, 1957.)

439*t*. Dorothy Maharam: *The realization of point-isomorphisms.*

It is shown that any set-isomorphism of a product of (perhaps uncountably many) normal measure spaces can be realized by a point-isomorphism. (The terminology is that of Halmos and von Neumann, Ann. of Math. vol. 43 (1942) pp. 332–350). (Received February 12, 1957.)

440*t*. J. L. Massera and J. J. Schäffer: *Linear differential equations and functional analysis.* I.

Let X, $\tilde{X}$ be a Banach space and its space of endomorphisms. The following spaces of functions of $t \in J = [0, +\infty)$ in X (or $\tilde{X}$) are considered: $\mathfrak{L}$, space of functions integrable in any finite interval; $\mathcal{C}$ space of bounded continuous functions with norm $|||x||| = \sup \|x(t)\|$; $\mathcal{C}_0$, $\mathcal{A}$, $\mathcal{P}_\omega \subset \mathcal{C}$, subspaces of functions $\to 0$ when $t \to +\infty$, almost-periodic, limit-periodic of period ω, respectively; $\mathfrak{M}$ space of functions $\in \mathfrak{L}$ with $\sup \int_t^{t+1} \|x(\tau)\| d\tau < \infty$ and norm $|||x|||_{\mathfrak{M}}$ equal to that sup; $\mathfrak{L}^p$, $1 \leqq p \leqq \infty$, space of pth-power integrable or essentially bounded functions with the usual norm. Assume that the values for $t = 0$ of the bounded solutions of (1): $\dot{x} + A(t)x = 0$, $A \in \mathfrak{L}(\tilde{X})$, $x \in X$, form a subspace X_0 (which is not always the case) and that X is the direct sum of X_0 and another subspace X_1. Theorem 1: If (2): $\dot{x} + A(t)x = f(t)$ has at least one bounded solution for each $f \in \mathfrak{F}$, $\mathfrak{F}$ being any one of $\mathcal{C}$, $\mathcal{C}_0$, $\mathcal{A}$, $\mathcal{P}_\omega$, $\mathfrak{M}$, $\mathfrak{L}^p$ $(1 \leqq p \leqq \infty)$, there exists K depending only on A and $\mathfrak{F}$, such that the (unique) bounded solution of (2) with $x(0) \in X_1$ satisfies $|||x||| \leqq K \cdot |||f|||_{\mathfrak{F}}$. Furthermore, if the assumption holds for any one of $\mathcal{C}$, $\mathcal{C}_0$, $\mathcal{A}$, $\mathcal{P}_\omega$ and if X_1 is reflexive it also holds for $\mathfrak{L}^\infty$. (Received January 28, 1957.)

441t. J. L. Massera and J. J. Schäffer: *Linear differential equations and functional analysis*. II.

With the notations of Part I, the following theorems are proved: 2. If for some p, $1<p\leqq\infty$, and each $f\in\mathcal{L}^p$, (2) has at least one bounded solution, there is a constant $\mu>0$ and a function $M(t_0)$, $t_0\in J$, such that for any bounded solution of (1) $\|x(t)\|\leqq M(t_0)\cdot\exp[-\mu(t-t_0)^{1/q}]\cdot\|x(t_0)\|$, $t\geqq t_0$, $(1/p)+(1/q)=1$. 3. The same result holds for $f\in\mathcal{C}$, $\mathcal{C}_0$ (and, if X_1 is reflexive, for $f\in\mathcal{A}$, $\mathcal{P}_\omega$) with $q=1$. 4. If moreover $A\in\mathfrak{M}$, positive constants N, N', ν, ν' exist such that for $t\geqq t_0$: (i) any bounded solution of (2) satisfies $\|x(t)\|\leqq N\exp[-\nu(t-t_0)]\cdot\|x(t_0)\|$; (ii) any solution with $x(0)\in X_1$ satisfies $\|x(t)\|\geqq N'\exp[\nu'(t-t_0)]\cdot\|x(t_0)\|$; (iii) any pair of solutions belonging to these two different kinds stay angularly apart (in a specified sense). 5. Conversely, (i), (ii), (iii) imply that (2) has at least one bounded solution for every $f\in\mathfrak{M}$, a fortiori for $f\in\mathcal{L}^p$ $(1\leqq p\leqq\infty)$, $\mathcal{C}$, $\mathcal{C}_0$, $\mathcal{A}$, $\mathcal{P}_\omega$; in this converse, (iii) may be replaced by $A\in\mathfrak{M}$. 6. If for every $f\in\mathcal{L}^1$ (2) has at least one bounded solution, (i), (ii), (iii) hold with $\nu=\nu'=0$ (no assumption is made on A). 7. The converse of Theorem 6 holds. (Received January 28, 1957.)

442t. J. L. Massera and J. J. Schäffer: *Linear differential equations and functional analysis*. III.

With the notations of the preceding Parts, the following theorems are proved: 8. If $A\in\mathfrak{M}$ and if a scalar function $V(x,t)$ exists with an infinitely small upper bound (nothing is assumed concerning the sign of V) such that its total derivative V' along the solutions of (1) is negative definite, all bounded solutions of (1) satisfy (i) and their initial values $x(0)$ form a subspace X_0; if X_1' is another subspace on which $V(x,0)\leqq 0$, all solutions of (1) with $x(0)\in X_1'$ satisfy (ii). 9. The same is true if instead of $V\leqq 0$ on X_1' we assume $X_0\cap X_1'=\{0\}$ and $\dim X_1'<\infty$ (counterexamples are given when these assumptions do not hold). 10. If the assumptions (i), (ii), (iii) are satisfied, given any even integer κ, functions $V_0(x,t)$, $V_1(x,t)$ exist which are homogeneous polynomials in x of degree κ, positive semidefinite and bounded $(0\leqq V_i\leqq S\cdot\|x\|^\kappa,\ i=0,1)$, such that V_0+V_1 is positive definite $(V_0+V_1\geqq s\cdot\|x\|^\kappa,\ s>0)$ and $V=V_0-V_1$ has a negative definite V'. (Received January 28, 1957.)

443t. J. L. Massera and J. J. Schäffer: *Linear differential equations and functional analysis*. IV.

Previous results of Parts I–III yield the following applications: 11. If the assumptions of Theorem 1 are satisfied and if $h(x,t)$ is any (nonlinear) function defined for $t\in J$, $\|x\|\leqq a$, such that $h(0,t)\in\mathfrak{F}$ and that for any two x', $x''\in X$ one has $\|h(x',t)-h(x'',t)\|\leqq\gamma_2(t)\|x'-x''\|$; where the scalar function $\gamma_2(t)\in\mathfrak{F}$, and if the norms of $h(0,t)$ and γ_2 in $\mathfrak{F}$ are sufficiently small, then the equation (3): $\dot{x}+A(t)x=h(x,t)$ has a family of bounded solutions $x(t,\xi_0)$, where $\xi_0\equiv x(0,\xi_0)$ takes any arbitrary value in a sphere $\xi_0\in X_0$, $\|\xi_0\|\leqq b$. 12. If the assumptions of Theorem 1 are satisfied with $\mathfrak{F}=\mathcal{C}_0$, then all bounded solutions of (2) tend to 0 when $t\to+\infty$; the same is true of the solutions of (3) described in Theorem 11 if the assumptions of this theorem are satisfied. 13. If $A\in\mathcal{A}[\mathcal{P}_\omega]$, if the assumptions of Theorem 1 are satisfied with $\mathfrak{F}=\mathcal{A}[\mathcal{P}_\omega]$ and if X_1 is reflexive, then, for each $f\in\mathcal{A}[\mathcal{P}_\omega]$, equation (2) has one and only one almost-periodic [limit-periodic of period ω] solution; the same is true of (3) if h satisfies the assumptions of Theorem 11. 14. The properties stated before (particularly in Theorems 4 and 13) are rough in the following sense: the set of operators $A \in \mathfrak{F}$ (where $\mathfrak{F} = \mathfrak{M}, \mathfrak{A}, \mathfrak{P}_\omega$ as the case may be) for which these properties hold is open in $\mathfrak{F}$. (Received January 28, 1957.)

444*t*. L. M. Milne-Thomson: *Gauss's theorem.*

By suitably defining a content-tensor for a parallelotope (generalized parallelpiped) in Euclidean space of n dimensions, Gauss's theorem for converting a volume into a surface integral is generalized to n-space. It then appears that the general theorem includes as special cases not only Gauss's but also Stokes's theorem, and the ordinary definite integral in one variable. (Received February 18, 1957.)

445. John Nash: *A priori Hölder continuity of solutions of parabolic and elliptic equations.*

Consider a heat equation $\nabla \cdot (C \cdot \nabla T) = \dot{T}$ where C is the (symmetric) conductivity tensor. C varies with position and time. Assume all eigenvalues e_ν of C everywhere obey $0 < c_1 \leqq e_\nu \leqq c_2$. Without continuity requirements on C, we have $|T(x_1, t) - T(x_2, t)| \leqq Bk|x_1 - x_2|^\alpha t^{-\alpha/2}$ if $|T| \leqq B$ everywhere when $t = 0$. The constants k and α depend only on c_1, c_2 and the dimension n of the space. For elliptic boundary value problems a Hölder exponent $\alpha/1+\alpha$ is derivable. The method depends on first estimating the moment $M(t) = \int rTdV$ of a fundamental solution, r being distance from its point of origin. This is done by interrelating M and $\dot{M}$ and the strictly increasing entropy $S(t) = -\int T \log T dv$ and $\dot{S}$. Result: $at^{1/2} \leqq M(t) \leqq bt^{1/2}$. Second, one shows that two adjacent fundamental solutions T_1 and T_2 overlap after a certain time so that $\int \min(T_1, T_2)dV \geqq \epsilon$ when $t \geqq \tau$. Third, one shows that $\int |T_1 - T_2| dV \to 0$ as $t \to \infty$ like $t^{-\alpha/2}$, which gives the result. These a priori estimates are of sufficient strength to apply to nonlinear equations. We believe the results and methods will open up areas of this field heretofore intractable, in particular the existence problems for compressible viscous fluid flows. (Received April 2, 1957.)

446*t*. O. G. Owens: *The explicit solution with given integral values of the reduced wave equation.*

The main motivation for this paper is in the sequence of steps which yields the Poisson integral solution of the classical boundary-value problem for Laplace's equation and the circular domain. The complete analogue of this process is developed for the two-dimensions reduced wave equation. The principal theorem obtained is the following: Assume that $u(x, y) \in C^2$ and satisfies the reduced wave equation, $u_{xx} + u_{yy} + u = 0$, for $x^2 + y^2 < \infty$. Furthermore, assume that uniformly in θ the integral $\int_0^\infty u(r, \theta)dr = F(\theta)$ $(x = r\cos\theta, y = r\sin\theta)$, $F(\theta)$ being of period 2π and fulfilling a uniform Hölder condition of order $\alpha > 1/2$. Then the integral-value problem has a unique solution with the integral representation $u(x,y) = (1/2\pi)\int_0^{2\pi} F(\phi)\cos r \sin(\phi - \theta)d\phi + (1/2\pi)^2 \int_0^{2\pi} d\phi F(\phi) \cdot \int_0^{\pi/2}([\sin r \sin(\tau + \phi - \theta) + \sin r \sin(\tau - \phi + \theta)]d\tau/\sin\tau)$. In the proof of this theorem an integral summation formula is derived for the sum $E(r, \theta) = J_0(r)/2 + \sum_{k=1}^\infty J_k(r)\cos k\theta$; namely, $E(r, \theta) = \cos r \sin\theta/2 + (1/2\pi)\int_0^{\pi/2}([\sin r \sin(\tau + \theta) + \sin r \sin(\tau - \theta)]d\tau/\sin\tau)$. (Received February 18, 1957.)

447. Ivan Polonsky: *Pseudoanalytic functions with characteristic coefficients in L_p.*

Let F and G be bounded functions with strong L_p $(p > 2)$ and L_{p_1} $(1 < p_1 < 2)$ derivatives as defined by Friedrichs and Sobolev. The characteristic coefficients a, b, A, B (cf. Bers, Bull. Amer. Math. Soc. vol. 62 (1956) p. 293) of the generating pair

(F, G) are then in $L_p \cap L_{p_1}$ ("condition C"). In a bounded domain a function w is (F, G) pseudoanalytic if and only if it has strong derivatives satisfying the equation $w_{\bar{z}} = aw + b\bar{w}$. Also if a, b satisfy "condition C" (i) there exists a generating pair (F, G) with a, b as the first two characteristic coefficients. (ii) (F, G) has a successor with characteristic coefficients satisfying "condition C." (iii) A bounded continuous function w is (F, G) pseudoanalytic if and only if $(x) f = w + (1/\pi)\int\int((aw + b\bar{w})/(\zeta - z))d\xi d\eta$ is analytic. The similarity principle of L. Bers is also proved under this weaker hypothesis and is then used to prove Taylor and Laurent expansion theorems for pseudoanalytic functions. By assuming that $a(z)|z|^{2(p-2)/p}$, $b(z)|z|^{2(p-2)/p}$ are also in L_p a theorem of Vekua about the existence of a solution of the equation (x) is extended to arbitrary domains. (Received February 19, 1957.)

448. R. T. Prosser: *Subalgebras of C^* algebras.* Preliminary report.

Let A be a C^* algebra with unit, and A' the dual space of bounded linear functionals on A. If B is a proper C^* subalgebra of A, also with unit, and $B^\perp$ the annihilator of B in A', then the Krein-Milman theorem implies that $B^\perp$, and hence B, is determined by the hermitian extreme points of its unit sphere. A careful analysis shows that every such extreme point may be uniquely resolved as half the difference of two positive extreme points (extreme states) of the unit sphere of A'. Thus every subalgebra of A, proper or not, is determined by the pairs of extreme states which it identifies. In particular, if two subalgebras identify the same pairs of extreme states, they must be the same; if a subalgebra separates all pairs of extreme points, then it must be A. Utilizing the correspondence between extreme states and maximal left ideals of A, we may rephrase these conclusions as follows: Every subalgebra of A is determined by its intersections with the maximal left ideals of A. In particular, if two subalgebras distinguish the same pairs of maximal left ideals, they must be the same; if a subalgebra distinguishes all pairs of maximal left ideals, it must be A. (Received February 20, 1957.)

449t. M. H. Protter: *Lower bounds for the first eigenvalue of elliptic equations.*

New estimates are obtained for the first eigenvalue of elliptic equations. In the simplest case of $u_{xx} + u_{yy} + \lambda^2 u = 0$ the expression $\int\int_D [u_x^2 + u_y^2 + 2Puu_x + 2Quu_y + (P_x + Q_y - \lambda^2)u^2]dxdy$ vanishes for arbitrary $P = P(x, y)$, $Q = Q(x, y)$ if $u = 0$ on the boundary of D. If P and Q are selected so that the above integrand, considered as a form in u_x, u_y, and u, is positive definite for a range of λ, $\theta \leqq \lambda \leqq \lambda_1$, then obviously λ_1, is a lower bound for the first eigenvalue. This can be done by taking P and Q as solutions of certain Riccati equations. The solution of these Riccati equations are in turn reduced to the solution of an eigenvalue problem for a linear second order ordinary differential equation. For some configurations, e.g. rhombus, this yields improved estimates over those obtained previously. The method can be extended by first applying Green's theorem to the expression $\int\int aLudxdy$ where A is an arbitrary positive function. (Received February 6, 1957.)

450t. Valdemars Punga: *On some property of a function satisfying Poisson's equation.*

It is known that if $\psi(x, y)$ is harmonic in a circular region with radius r, then the value of ψ at the center 0 is $\psi_0 = (1/2\pi)\int_0^{2\pi}\psi_r d\theta$, where ψ_r is the value of ψ on the circumference C. The finite-difference equivalent of this theorem can be stated in the following form: subdividing C into n equal arcs at points $1, 2, \cdots, n$, $\psi_0=(1/n)\cdot\sum_{i=1}^n \psi_i+R(r^n)$, where error $R(r^n)$ is of order r^n. It is shown that if ψ satisfies Poisson's equation $\nabla^2\psi=f(x, y)$, then, for n odd ($n\geqq 3$): $\psi_0=(1/n)\sum_{i=1}^n \psi_i-(r^2/4)f_0-(r^4/64)\nabla^2 f_0-\cdots-(r^{n-1}/2^{n-1}[((n-1)/2)!]^2)\nabla^{n-3}f_0+R(r^n)$, where f_0 is the value of f at 0, and for n even ($n\geqq 4$): $\psi_0=(1/n)\sum_{i=1}^n \psi_i-(r^2/4)f_0-(r^4/64)\nabla^2 f_0-\cdots-(r^{n-2}/2^{n-2}[((n-2)/2)!]^2)\Delta^{n-4}f_0+R(r^n)$. (Received February 19, 1957.)

451*t*. R. A. Raimi: *On compactness in a space of operators*. Preliminary report.

Let H be an infinite dimensional Hilbert space, $L(H, H)$ its algebra of bounded linear transformations. Dixmier (Ann. Math. vol. 51 (1950)) discusses these topologies on L: L_s the "strong" topology of operators, L_s^* the topology transported from L_s by the mapping $T\leftrightarrow T^*$ (adjoint), and ^+L the common refinement of L_s and L_s^*. The present paper gives criteria for compact subsets in these topologies, and shows them to be distinct. E.g., A has compact closure in L_s^* if and only if it is equicontinuous as a set of mappings from H_k to H_k, where H_k is H fitted with the compact-open topology when considered as a set of functionals on H. A has compact closure in L_s if and only if $\{T^* \mid T\in A\}$ has the same equicontinuity property. Compactness in ^+L is equivalent to possession of both properties. It follows that in any of the three topologies, the closed convex circled hull of a compact set is again compact. Finally, since for norm-bounded subsets of $L(H, H)$ the "strong" topologies are equivalent to the "strongest" topologies (op. cit.), the same criteria apply there too. (Received February 22, 1957.)

452*t*. R. A. Raimi: *Mean values and Banach limits*, II.

In a previous paper (Bull. Amer. Math. Soc. Abstract 64-4-512) a class of Banach limits was obtained, using integral means, for translation-invariant Banach spaces E of real measurable bounded functions on $R(-\infty, \infty)$, and their extreme values computed. Here the line is replaced by any abelian group G, and convex combinations of the translation operators $T_s: f(x)\rightarrow f(x+s)$ replace the integral means. By the use of a fixed point theorem closely related to Eberlein's formulation of the mean ergodic theorem (Trans. Amer. Math. Soc. vol. 67 (1949) pp. 217–240), all Banach limits for the space of bounded functions on G are constructed, and their extreme values found to be lim sup and lim inf of $T_\sigma f(0)$, where T_σ is a convex combination of translation operators, and the set $\{T_\sigma\}$ is made a net in the ordering given by Eberlein to a bounded abelian semigroup. It follows, via an example, that there are (nonuniformly) continuous functions on R which have Banach limits not given by the integral mean construction. Other kinds of invariant finitely additive measures can be found by the same method. (Received February 22, 1957.)

453*t*. I. F. Ritter: *The multiplicity function of an equation.*

The *multiplicity function* $\mu(z)=[1-ff''(f')^{-2}]^{-1}$ is extremely helpful in solving the equation $f(z)=0$. For any root r, Lim $\mu(z)=m$; $z\rightarrow r$, determines the multiplicity m of that root. When $m\neq 1$, or when roots with $m=1$ are not widely separated, $|1-\mu^{-1}(z)|$ will be, or is apt to be, far from zero, even if z is near r. This accounts for the unsatisfactory rate of convergence, under those conditions, of the Newton-Raphson iteration and of its modifications based on reversion of Taylor's series. The magnitude of $\mu(z)$ shows that simple roots need not be very close together to produce this retardation. With the guidance furnished by the behavior of $\mu(z)$, this trouble is overcome, and the iteration formula $\phi(z)=z-[\mu(z)+m]f(z)/2f'(z)$ can be exploited to provide *third* order convergence to any root independently of its multiplicity or of the proximity of other roots. When formal differentiation of $f'(z)$ is to be avoided, a Hermitian type interpolation formula can be used to approximate values of $f''(z)$. This hardly impairs the effectiveness of the proposed method for dealing with multiple or nearly coincident roots. (Received January 21, 1957.)

454*t*. G. C. Rota: *A spectral mapping theorem for the essential spectra of ordinary linear differential operators.*

An ordinary differential operator $\tau=a_n(t)(d/dt)^n+a_{n-1}(t)(d/dt)^{n-1}+\cdots+a_0(t)$ defined on an interval $I=(a, b)$ $(-\infty\leqq a<a\leqq\infty,\ a_n(t)>0,\ a_i$ of class C^∞, $0\leqq i\leqq n)$ acting on the set of functions with compact carriers in I defines a linear operator in each of the B-spaces $L_p(I)$ $(1\leqq p\leqq\infty)$ whose closure $T_0(\tau, p)$ exists. The adjoint of the operator $T_0(\tau, p)$ is calculated. The essential spectrum of τ in $L_p(I)$ is defined as the set of all complex numbers λ for which the range of the operator $\lambda I-T_0(\tau, p)$ is not closed. The essential spectrum of τ is a closed set and coincides with the essential spectrum of the operator $T_0(\tau^*, q)$, where τ^* is the Lagrange adjoint of τ and $1/p+1/q=1$. *Theorem*: Let A be the essential spectrum of τ in $L_p(I)$ and let $p(z)=b_nz^n+\cdots+b_0$ be a complex polynomial. Then the essential spectrum of the operator $p(\tau)=b_n\tau^n+\cdots+b_0$ is the set $\{p(\lambda)\,|\,\lambda\in A\}$. (Received January 29, 1957.)

455*t*. G. C. Rota: *Extension theory of non-self-adjoint differential operators in B-spaces.*

An extension theory for ordinary linear differential operators in the spaces $L_p(I)$ (I an arbitrary interval of the line, cf. the preceding abstract) can be constructed which generalizes several features of von Neumann's extension theory of symmetric unbounded operators in Hilbert space. Some of the main results are the following: Let C be a connected component of the complement of the essential spectrum of the differential operator τ in $L_p(I)$. For every complex number λ in C the dimension of the space of solutions of class $L_p(I)$ of the differential equation $(\lambda-\tau)f=0$ is constant; this dimension is the deficiency index of the component C. An extension T of the operator τ in $L_p(I)$ is a closed operator such that $T_0(\tau, p)\subset T\subset T_0(\tau^*, q)^*$ $(1/p+1/q=1)$. *Theorem*: Assume that the deficiency indices of all connected components of the essential spectrum of τ are equal. Then there exists an extension T of τ whose spectrum consists of the essential spectrum of τ and a set of eigenvalues all of whose limit points lie in the essential spectrum of τ. A suitably stated converse of the theorem is also true. Explicit representations of extensions and their resolvents are derived, as well as several explicit results on the location of eigenvalues. (Received January 29, 1957.)

456. Robert Schatten: *On the space of completely continuous operators.*

Let $\mathcal{C}$ be the space of all completely continuous operators on a Hilbert space $\mathcal{H}$. Thus, its conjugate space $\mathcal{C}^*$ is the "trace-class" and $\mathcal{C}^{**}$ the space of all bounded operators on $\mathcal{H}$. Put $\mathcal{C}^*=\mathcal{C}^{(1)}$ and $(\mathcal{C}^{(n-1)})^*=\mathcal{C}^{(n)}$ for $n>1$. Since the "natural" imbedding of $\mathcal{C}$ into $\mathcal{C}^{(2)}$ is a proper one, it follows that $\mathcal{C}\subset\mathcal{C}^{(2)}\subset\mathcal{C}^{(4)}\subset\cdots$ and $\mathcal{C}^{(1)}\subset\mathcal{C}^{(3)}\subset\mathcal{C}^{(5)}\subset\cdots$ where each inclusion is a proper one. It is shown that $\mathcal{C}$ is indeed a "beginning" that is, $\mathcal{C}$ is not the conjugate space of any Banach space. (Received November 28, 1956.)

457. Martin Schechter: *On estimating partial differential operators. II.*

Let G be a bounded domain in E^n with closure $\overline{G}$ and boundary B of class C^m. At each point $x \in B$ assign a nonnegative integer $r(x) \leqq m$ and denote by $C_{mr}(G)$ the set of all (complex) functions u of class C^m in $\overline{G}$ having all derivatives of order $< r(x)$ vanish on B near x for each point $x \in B$. Let (A_k) be a finite set of linear mth order partial differential operators with complex valued coefficients continuous in $\overline{G}$. Consider the complex vector roots common to the characteristic polynomials P_k of the A_k. At each point $x \in B$ let $q(x)$ be the greatest number of such roots (properly defined) with imaginary parts perpendicular to B at x and pointing in the same direction (inward or outward). Let $\|u\|$ denote the $L_2(G)$ norm of u and $D^m u$ its generic mth order derivative. Then there is a constant K such that $\|D^m u\|^2 \leqq K(\sum \|A_k u\|^2 + \|u\|^2)$ for all $u \in C_{mr}(G)$ if and only if the P_k have no real vector roots in common and $q(x) \leqq r(x)$ at each point $x \in B$. (Received February 1, 1957.)

458. Annette Sinclair: *A general solution for a class of approximation problems.*

This paper is concerned with a class of approximation problems of which Carleman's approximation theorem and the author's generalization of Runge's Theorem are examples. In proofs of theorems of this type a nested sequence of sets $\{R_i\}$, depending on a given set $S \subset R$, is constructed. Then a sequence of functions $\{r_i(z)\}$ is defined in such a way that each $r_i(z)$ meets specified conditions on $R_n \cap S$ and such that $\lim r_i(z)$ satisfies preassigned conditions on S and belongs to a designated class of functions defined on all of R. In this paper a theorem abstracting this type of problem is proved and is applied to obtain some new results on approximation by analytic functions. (Received February 18, 1957.)

459. Paul Weiss: *Sampling theorems associated with Sturm-Liouville systems.*

A class C of functions $f(t)$ of the continuous variable t, $-\infty < t < \infty$, is said to satisfy a sampling theorem if there exists a discrete set of values t_n and a kernel $K(t, t_n)$ with the property $K(t_n, t_m) = \delta_{mn}$ such that $f(t) = \sum_n K(t, t_n) f(t_n)$ for every function $f(t)$ of C. It is shown that a sampling theorem is associated with each positive Sturm-Liouville system (one with positive eigenvalues), the class C and the kernel K depending on the system. Let $u(x, t)$ be the solution of the Sturm-Liouville differential equation $(p(x)u_x)_x + (-q(x) + t^2\rho(x))u = 0$, $a \leqq x \leqq b$, $p(x) > 0$, $\rho(x) > 0$, $q(x) \geqq 0$, satisfying the boundary conditions at $x = a$: $u(a, t) = \sin \alpha$, $p(a)u_x(a, t) = \cos \alpha$, $0 \leqq \alpha \leqq \pi/2$, so that $l_\alpha[u(a, t)] \equiv \cos \alpha \cdot u(a, t) - \sin \alpha \cdot p(a) \cdot u_x(a, t) = 0$. Let t_n^2 be the set of eigenvalues and $u(x, t_n)$ be the set of corresponding eigenfunctions obtained by imposing the additional boundary condition at $x = b$: $l_\beta[u(b, t)] \equiv \cos \beta \cdot u(b, t) + \sin \beta \cdot p(b) \cdot u_x(b, t) = 0$, $0 \leqq \beta < \pi/2$. Let an even function $h(t)$ be called "representable" if it can be expressed in the form $h(t) = \int_a^b g(x) u(x, t) \rho(x) dx$, where $g(x)$ satisfies $l_\alpha[g(a)] = 0$, $l_\beta[g(b)] = 0$, and possesses a Fourier expansion in terms of the eigenfunctions $u(x, t_n)$. Denote by $f_e(t)$ and $f_0(t)$ respectively the even and odd parts of an arbitrary function $f(t)$. Then a sampling theorem holds for the class G of functions $f(t)$, for which $f_e(t)$ and $t \cdot f_0(t)$ are representable. It reads $f(t) = \sum_{n=-\infty}^{\prime\infty} K(t, t_n) f(t_n)$, where the kernel is given by $K(t, t_n) = (t_n/t)(1/(t - t_n))(l_\beta[u(b, t)]/l_\beta[u_t(b, t_n)])$ in the sum, $t_{-n} = t_n$, and the prime denotes that the term with $n = 0$ is to be omitted. (Received February 21, 1957.)

460. Burton Wendroff: *An integral inequality.*

The inequality $u(x) \leqq C + \int_{x_0}^{x} v(s)u(s)ds$, with $u(x) \geqq 0$, $v(x) \geqq 0$ for $x \geqq x_0$, has the solution $u(x) \leqq C \exp\left[\int_{x_0}^{x} v(s)ds\right]$. A partial generalization to functions of two independent variables can be obtained; namely, let $u(x, y) \leqq a(x) + b(y) + A\int_{x_0}^{x} u(r, y)dr + B\int_{y_0}^{y} u(x, s)ds + \int_{x_0}^{x}\int_{y_0}^{y} v(r, s)u(r, s)drds$ where (a) $u(x, y)$, $v(x, y)$ are continuous, and (b) $u(x, y) \geqq 0$, $v(x, y) \geqq 0$, $a(x) \geqq 0$, $a'(x) \geqq 0$, $b(y) \geqq 0$, $b'(y) \geqq 0$, for $x_0 \leqq x$, $y_0 \leqq y$. This is solved by showing that $u(x, y)$ is majorized by the solution of the corresponding equality, and then obtaining an estimate for that solution. The final result is $u(x, y) \leqq (Q(x_0, y)Q(x, y_0)/Q(x_0, y_0)) \exp\left\{A(x - x_0) + B(y - y_0) + \int_{x_0}^{x}\int_{y_0}^{y} [AB + v(r, s)]drds\right\}$ where, $Q(x_0, y_0) = a(x_0) + b(x_0)$, $Q(x_0, y) = a(x_0) + b(y_0) + \int_{y_0}^{y} e^{-B(y' - y_0)} b'(y')dy'$, $Q(x, y_0) = a(x_0) + b(y_0) + \int_{x_0}^{x} e^{-A(x' - x_0)} a'(x')dx'$. (Received February 22, 1957.)

461. Albert Wilansky: *Some Tauberian lemmas.*

Consider conservative matrices $A = (a_{nk})$ with $a_{nk} = 0$ for $k > n$ called triangular. If also $a_{nn} \neq 0$, call A a triangle. Let $a_k = \lim_n a_{nk}$, $\|A\| = \sup_n \sum_k |a_{nk}|$. THEOREM 1. *If $a_{nn} = 1$ for all n and $\sum |a_k| > \|A\| - 2$ then $\|A^{-1}\| < \infty$, i.e. Ax convergent implies x convergent.* The special case in which A is regular is known. Examples show this best possible. There follows the known result: $\{x_n + \sum_{k=1}^{n-1} t_k x_k\}$ convergent, $\sum |t_k| < \infty$, implies x convergent. Call A right-finite if there exists m such that $a_{nk} = 0$ for $k > m$. If A, B are conservative triangles with $\|B^{-1}\| < \infty$ and $A - B$ right-finite, clearly $\|A^{-1}\| < \infty$. The result is extended by omitting "conservative." This leads (with a remark of K. Zeller) to THEOREM 2. *The Banach algebra U of triangular matrices A with $\|A^{1}\| < \infty$, is not semi-simple.* THEOREM 3. *There exist bounded sequences x, y, y divergent such that $x - Ay$ is divergent for every triangular conservative A.* This contrasts with the known result that there exists conservative A such that $x - Ay = 0$; and says the map $f: U \to m/c$ is not onto. Call A generalized conservative if Ax is in the linear closure in c of the columns and the row-sum of A for all $x \in c$; generalized coercive if the same for all $x \in m$. This allows extension of theory of conservative matrices to class of matrices of finite norm. (Received February 18, 1957.)

462. František Wolf: *The singular spectrum of partial differential operators.*

In the case of a partial differential elliptic operator A in the Hilbert space $L^{\gamma}(D)$, suppose $\lambda \in \sigma_e(A)$ the essential spectrum of A. Then there exists a normalized noncompact sequence of $f_n \in L^2(D)$ such that $\lim (A - \lambda)f_n = 0$. If λ is not an eigenvalue of A, then f_n converges to zero uniformly in every interior domain and at all "regular" points near the boundary. The irregular points are those in whose every neighborhood the noncompactness persists. Those points cause the essential spectrum at λ. We may write $\lambda \in \sigma_e(x)$, if x is such a point. If $\lambda \notin \sigma_e(x)$, for all $x \in \bar{D}$, then $\lambda \notin \sigma_e(A)$. In the case of boundary conditions of local character, we may say that $\sigma_e(A)$ derives from certain specific irregular points on the boundary. It can be shown that points which are regular for the differential equation and in whose neighborhood are valid boundary conditions of most of the usual types, are regular points. (Received February 22, 1957.)

APPLIED MATHEMATICS

463. J. H. Blau: *Social choice functions and simple games.*

A (social) *choice function* is a mapping from n-ples of strong orderings of a set A

to reflexive connected relations on A. It is *binary* if on each subset of A it induces a unique choice function. If and only if the function is *neutral* (Blau, *Econometrica* vol. 25 (1957)), it may be identified with a simple game, designating the *dictatorial* sets (ibid.) as winning coalitions. A *bicyclic ballot* is an n-ple on a three-element subset of A, in which exactly two distinct strong orderings occur, these being cyclically equivalent. A choice function is *bicyclically transitive* if the social order is transitive for those ordered triples which occur as individual strong orderings in a bicyclic ballot. A choice function satisfies the *weak unanimity rule* (URR) if a unanimous vote $x>y$ implies $x\geqq y$ socially. *Theorem*: Let A have more than two elements. A binary choice function with URR is a simple game if and only if it is monotonic (ibid.) and bicyclically transitive. Other results are obtained without URR or monotonicity, and with a reduced domain. (Received February 18, 1957.)

464. R. J. Duffin and E. P. Shelly (p): *Difference equations of polyharmonic type.*

Of concern are partial difference equations with constant coefficients. Fourier methods are used to study growth rate theorems and properties of the fundamental solution. A study is made of polynomial solutions and continuation theorems. As is well known, there are differential operators such as $r\cdot\text{grad}$ which convert harmonic functions into harmonic functions. Analogous operators are developed for the theory of discrete harmonic functions. By the use of such operators, it is found possible to give a complete evaluation of the fundamental solution of the Laplace difference equation in three dimensions. If f is harmonic, then r^2f is biharmonic. Relationships of this type are extended to the discrete case, and such relationships leads to a complete evaluation of the fundamental solution of the discrete biharmonic equation in two dimensions. (Received February 18, 1957.)

465*t*. J. R. Foote: *Recursion relations for parabolic cylinder functions.*

Formulas are found permitting the raising or lowering of the index ν of the parabolic cylinder function $D_\nu(z)$ appearing in certain derivatives and indefinite integrals. The functions are various combinations of the D_ν and of $\exp[-z^2/4]$, such as: $z^nD_\mu D_\nu$, $e^{-z^2/2}D_\mu D_\nu$, $e^{-3z^2/4}D_\mu D_\nu$, $e^{\pm z^2/4}D_\nu$, where we may have $n=0$, and/or $\mu=\nu$. When μ, ν, n are integers, the recursion relations allow reduction of the derivatives and integrals to algebraic calculations among similar functions and to integrals related to the error function. (Received February 19, 1957.)

466*t*. J. R. Foote: *An asymptotic method for free convection past a vertical plate.*

Recently Finston noted that there is a similarity transformation of the free convection equations which requires a variable plate temperature. The resulting pair of nonlinear equations may be solved in an iterative manner by a method recently detailed by Fettis which gives a system of linear equations with constant coefficients in this case. Three terms of the series are given for arbitrary values of three parameters. (Received February 19, 1957.)

467. Abolghassem Ghaffari: *On the solutions of compressible flow past a wedge.*

Applying the hodograph method (Temple & Yarwood: *Compressible flow past a* *wedge*, Report No. S.M.E. 3222) for the regions $q>q_1$ and $q>q_1$ (q is the incompressible fluid speed around the wedge and q_1 its value in the main stream), the solutions for the incompressible field of flow and the corresponding compressible field of flow around a wedge are obtained. It is found that the solution for the stream function ψ of a steady, irrotational compressible flow is either (I) $\psi = lq_1 \sum_{n=0}^{\infty} (\psi_m(\tau)/\psi_m(\tau_1)) \cdot \sin((n+1)/a)\theta, m=(n+1)/a$, for $q<q_1$, or (II) $\psi = lq_1 \sum_{n=0}^{\infty} (\psi_{-m}(\tau)/\psi_{-m}(\tau_1)) \sin(n/a), \theta$, $m=n/a$, for $q>q_1$, where l is a fraction of the length L of the sloping part of the wedge, $a=\alpha/\pi$, where 2α is the wedge angle $(0<a<1)$, and θ is the angle between the x-axis and the direction of flow. $\tau_1 = w_1^2/w_m^2$ is the value of $\tau = w^2/w_m^2$ (where w is the compressible fluid speed, w_m its maximum, and w_1 the value of w in the main stream) in the main stream, $\psi_n(\tau) = \tau^{m/2} F_m(a_m, b_m, c_m; \tau)$ where $F_m(\tau)$ indicates the hypergeometric function. $\psi_{-m}(\tau)$ is some suitable solution of the equation for $\psi_m(\tau)$ in the region $\tau_1<\tau<1$. It is shown that the solutions (I), (II), are continuous at $\tau=\tau_1$, and that the continuity of $\partial\psi/\partial\tau$ at $\tau=\tau_1$ implies $[\psi'_m(\tau_1)/\psi_m(\tau_1)]_{m=(n+1)/a} = [\psi'_{-m}(\tau_1)/\psi_{-m}(\tau_1)]_{m=(n+1)/a}$. It is found that $\psi_{-m}(\tau)$ is bounded at $\tau=1$, and that when $w_m \to \infty$, $\psi_{-m}(\tau)/\psi_{-m}(\tau_1) \to (\tau/\tau_1)^{-m/2}$. (Received February 20, 1957.)

468. M. A. Hyman: *Eigenvalues and eigenvectors of general matrices.*

This paper extends to general matrices (complex elements, no symmetry) the method previously presented (Cambridge, October, 1956) for finding eigenvalues and eigenvectors or real matrices. By $(N-1)(N-2)/2$ complex rotations (unitary transformations each involving only two coordinate directions) the general matrix A of order N is transformed into matrix B with all zeros above the first super-diagonal: $B=R^{-1}AR$. The eigenvalue problem for B is then $(B-\lambda I)x=0$, where x is a column vector with components $x_1, x_2, \cdots, x_N$. Setting (usually) $x_1=1$ and guessing λ, one calculates $x_2, x_3, \cdots, x_N$ and $F(\lambda)$ recursively; $F(\lambda)$ is the characteristic polynomial, evaluated for the trial λ. By successive approximation (see abstract below) a sequence of λ's (usually complex) is obtained which approaches an eigenvalue E of B: $F(E)=0$. Simultaneously, the vectors x approach the eigenvector ξ. The corresponding eigenvalue and eigenvector of A are E, $R\xi$. Other λ sequences can be constructed which converge to the other eigenvalues of B; the x-vector sequences generated as a by-product of evaluating $F(\lambda)$ converge to eigenvectors of B. (Received February 21, 1957.)

469t. M. A. Hyman: *Finding complex zeros by interpolation*, II.

Consider $F(\lambda)$, a general function of the complex variable λ, with zero E. We have previously shown (Cambridge, October, 1956) how to construct a sequence $\{\lambda_i\}$ converging to E. Use is made of $\{P_i^{(N+1)}(\lambda)\}$, a sequence of Nth degree polynomials $(N=1, 2, 3, \cdots)$, coinciding with $F(\lambda)$ at $N+1$ points near E, and approximating $F(\lambda)$ ever more closely in that neighborhood. These "interpolation" methods for finding zeros appear to have many advantages for use with automatic computing machines. We here emphasize the use of "2-point" and "3-point" methods, with which we have had good success in numerical experiments. Let the respective interpolating polynomials be $P_{i+1}^{(2)}(\lambda) = \alpha_i(\lambda-\bar\lambda_i)+\beta_i$ and $P_{i+}^{(3)}, = A_i(\lambda-\bar\lambda_i)^2 + B_i(\lambda-\bar\lambda_i)+C_i$, where $\bar\lambda_i$ is the best approximation to E after i steps, and let the next approximation be $\lambda_{i+1}=\bar\lambda_i-\delta_i$; here $\delta_i=\beta_i/\alpha_i$ (2-point) and $\delta_i=C_i/B_i$ or $\delta_i=C_i/B_i+(A_i/B_i)(C_i/B_i)^2$ (3-point). $\beta_i=C_i=F(\bar\lambda_i)$; α_i, B_i are, respectively, the 2-point and 3-point approximations to $F'(\bar\lambda_i)$, if this derivative exists. Neither 3-point method (the second converges more rapidly) requires taking a square root. If λ_{i+1} is not closer to E (measured by the norm $|F(\lambda)|^2$) than one of the points from which it is obtained, δ_i is multiplied by K $(0<K<1)$ and λ_{i+1} recomputed. Any other zeros of $F(\lambda)$ are obtained by seeking zeros for $F(\lambda)|\prod_{r=1}^{n}(\lambda-E_r)$, where $E_1, E_2, \cdots, E_n$ are the roots already obtained (suggestion by G. E. Forsythe). (Received February 21, 1957.)

470. G. S. S. Ludford and S. H. Schot (p): *On sonic limit lines in the hodograph method.*

Lines in the physical plane on which the Jacobian $J=\partial(x, y)/\partial(q, \theta)$ of the hodograph transformation vanishes are called *limit lines*, of which two types have been distinguished. In a recent paper H. Geiringer (Math. Zeit. vol. 63 (1956) pp. 514–524) has identified a new type of limit line which coincides with a nondegenerate segment of the sonic line. She has called this a *sonic limit line*. In the present paper certain statements concerning the occurrence and geometry of sonic limit lines are proved, and a method of constructing flows with such lines is given, together with an example. This flow has no source-like character (cf. F. Ringleb, ZAMM vol. 20 (1940) pp. 185–198) and also contains an example—believed to be the first—of Geiringer's sonic double limit point. (This work was supported by the Office of Ordnance Research, U. S. Army, under Contract DA-36-034-ORD-1486 with the University of Maryland.) (Received February 19, 1957.)

471. Willard Miranker: *The reduced wave equation in a medium with a variable index of refraction.*

In this paper we discuss solutions u of the equation $\Delta u+h^2(x)u=0$ where $x=(x_1, x_2, x_3)$ and where $\lim_{|x|\to\infty} h^2(x)=k^2$. First we consider the case in which u is defined and bounded in all of space. We show that the L^2 norm of u over a sphere does not tend to zero as the radius of the sphere tends to infinity unless $u\equiv 0$. This result is an extension of one of Rellich who proved the same statement for a solution of $\Delta u+k^2u=0$. From our result we are able to demonstrate the nonexistence of quadratically integrable u for a suitable class of $h^2(x)$ which includes the case $h^2(x)\equiv k^2=$constant. Then we consider the exterior boundary value problem for $\Delta u+h^2(x)u=0$ subject to the radiation condition. We show that if $u\not\equiv 0$ there exists a solution $v\not\equiv 0$ of $\Delta v+k^2v=0$ such that $\lim_{|x|\to\infty}|u-v|=0$. From this osculation theorem we define the validity of the Rellich estimate for v and from this in turn the uniqueness theorem for the exterior boundary value problem for v. (Received February 27, 1957.)

472t. C. S. Morawetz: *Nonexistence of continuous transonic flows past profiles*, III. Preliminary report.

Suppose that for a given pressure-density relation there exists a smooth transonic flow past a profile with a subsonic speed q_∞ at infinity. By changing slightly the pressure-density relation for supersonic speeds a whole manifold of such profiles with smooth flows having the same speed at infinity can be shown to exist. For this set of flows the perturbation problem, which corresponds to finding a flow for a small change $(q_\infty+\delta q_\infty)$ of the speed at infinity, has no continuous solution. (Received February 20, 1957.)

473. L. E. Payne (p) and H. F. Weinberger: *Bounds for the displacement of free and supported plates.*

A method is given for finding arbitrarily close upper and lower bounds for the normal displacement at an interior point of a thin convex plate when the moment and either the shear or the normal displacement are prescribed on the boundary. The given quantities are approximated in a square integral norm by a linear combination of biharmonic functions, and the error in displacement is estimated in terms of this norm. The method is similar to that used by the authors for estimating the displacement of membranes and clamped plates (J. Math. and Phys. vol. 33 (1955) pp. 291–307). (Received February 21, 1957.)

474. I. F. Ritter: *Computation of the exact solution of a system of linear equations.*

If the numerical matrices A and R have integer elements, the solution X of the equation $AX=R$ is given by ratios of integers. A systematic way of computing these integers in sections of s digits on a calculator with place capacity s is described in form of a modification of the error correction in Bull. Amer. Math. Soc. Abstract 62-3-349. The desirability of obtaining the exact inverse of a matrix is indicated by J. Barkley Rosser [Jour. Res. Nat. Bur. Stand. vol. 49 (1952) p. 349], who also demonstrates a feasible scheme for computing it. The compact method of approximation by sections proposed here was found less laborious than Rosser's scheme, but its practical significance rests on its ability to avoid all those difficulties that one encounters in computing *decimal* approximations to X for a singular or ill-conditioned matrix A. (Received February 20, 1957.)

475. Charles Saltzer: *Multiple-valued solutions of the Laplace difference equation.*

By the use of the difference analogues of the Cauchy-Riemann equations multiple-valued solutions of the Laplace difference equation are defined and computed from the Green's function for the Laplace difference equation. The solution of the Dirichlet problem for the exterior of a finite slit for the case where the boundary values on opposite sides of the slit are different is discussed. (Received February 20, 1957.)

476. H. E. Salzer: *Note on multivariate interpolation for unequally spaced arguments, with an application to double summation.*

In two earlier articles the author expressed the multiple Gregory-Newton interpolation formula for $f(x, y, \cdots, z)$ in terms of mixed advancing differences as far as the nth order, in the simpler form $\sum\sum \cdots \sum_{i+j+\cdots+k=0,1,\cdots,n} C_{ij\cdots k}(x, y, \cdots, z) \cdot f(x_i, y_j, \cdots, z_k)$, where the function $C_{ij\cdots k}(x, y, \cdots, z) \equiv C_{ij\cdots k}$, the coefficient of the tabular entry $f(x_i, y_j, \cdots, z_k)$ was given explicitly, and even tabulated. This present note gives the generalization to where the independent variables are unequally spaced, by expressing that general mixed divided difference formula approximation to $f(x, y, \cdots, z)$ which is exact for an m-ary polynomial of total degree n, directly in terms of the tabular entries $f(x_i, y_j, \cdots, z_k)$. The new formula avoids the labor in the computation of the mixed divided differences $[x_0 \cdots x_r; y_0 \cdots y_s; \cdots; z_0 \cdots z_u]$ and is a simpler looking expression for the m-ary n-ic approximation to $f(x, y, \cdots, z)$. The main result is (1) $C_{ij\cdots k} = (x-x_i)(y-y_j) \cdots (z-z_k) \times \sum_{u=k}^{n-i-j-\cdots} L_k^{(u+1)}(z)/(z-z_u) \cdots \sum_{s=j}^{n-i-t-\cdots-u} L_j^{(s+1)}(y)/(y-y_s) \sum_{r=i}^{n-s-t-\cdots-u} L_i^{(r+1)} \cdot (x)/(x-x_r)$, where $L_j^{(s+1)}(y)$, for example, is the $(s+1)$-point Lagrangian coefficient at y_j for points $y_0, y_1, \cdots, y_s$. The coefficients $C_{ij\cdots k}(x, y, \cdots, z)$ satisfy $\sum\sum \cdots \sum_{i+j+\cdots+k=0,1,\cdots,n} (C_{ij\cdots k}(x, y, \cdots, z)=1$, $C_{ij\cdots k}(x_f, y_g, \cdots, z_h) = \delta_f^i \delta_g^j \cdots \delta_h^k$, and $\sum\sum \cdots \sum_{i+j+\cdots+k=0,1,\cdots,n} C_{ij\cdots k}(x, y, \cdots, z) x_i^0 y_j^p \cdots z_k^q$

$=x^0y^p \cdots z^q$ whenever $0+p+\cdots+q\leqq n$, which suggests the name "m-ary Lagrangian" coefficients. From (1), new formulas were derived for determining the limits of double sequences $S_{p,q}$ as $p, q\to\infty$, the arguments $x=1/p$ and $y=1/q$ being unequally spaced. (Received March 21, 1957.)

477. Joanna Wood-Schot: *On the numerical solution of Poisson's equation in an elliptic region.*

One difficulty in the numerical solution of elliptic differential equations in simple closed regions is the treatment of points near curved boundaries. This problem is investigated for Poisson's equation in an ellipse with given boundary conditions. Since second order difference approxinations are used at all interior points, truncation errors are introduced only at mesh boundary points, and hence their effect is studied by comparison with the analytic solution. The equation was solved on a high-speed computer using two iterative methods—the Gauss-Seidel method and the Successive Overrelaxation method (as developed by D. M. Young). The rate of convergence of the latter method is compared with that of both the Jacobi and the Gauss-Seidel methods. An error analysis of the results is given. (Received February 19, 1957.)

GEOMETRY

478. E. N. Gilbert (p) and H. O. Pollak: *A maximum-minimum problem connected with trees.*

Given N points in the plane, a *minimal tree* is a connected linear graph of shortest total line length having these N points as its vertices. If the N points are allowed to range over a closed bounded region R the lengths of minimal trees will somewhere attain a maximum, say $f(R, N)$. There exist numbers $K(R)$ and $C(R)$ such that $f(R, N) \leqq K(R)(N-1)^{1/2}$ and $C(R)(N-1)^{1/2}\leqq f(R, N)$. The upper bound is obtained using the following geometric property of minimal trees. If L is a line of a minimal tree let $D(L)$ denote the diamond-shaped figure which is the union of the two isosceles triangles having base line L and 30° base angles; then two such diamonds $D(L_1)$, $D(L_2)$ can intersect only at boundary points. The lower bound follows either by arranging N points in a regular lattice or by estimating the mean length of a minimal tree when N points are placed in R at random. If R is the unit square, suitable constants are $K(R)=2.34$, $C(R)=1$. (Received February 21, 1957.)

479. Donald Greenspan: *On two nonequivalent approaches to the study of vertices in E^3.*

In an attempt to generalize to E^n and to generalize topologically theorems on vertices, one must decide upon a definition of a vertex. For E^3, two possibilities readily suggest themselves. The first utilizes the concept of a stationary value of the radius of the osculating sphere, while the second makes use of the ideas of order of contact of a curve with an osculating sphere. It is shown that these are not equivalent and that one kind of vertex is a special type of the other kind. An example is provided to show that the radius of an osculating sphere may actually attain an extreme value without the curve and sphere having more than four consecutive points in common. (Received February 11, 1957.)

480. Melvin Hausner: *Differential representations of homotopy.*

Differentials $\alpha=\sum A_i dx^i$ are studied on a manifold M. The A_i are elements of some associative algebra A. It is then possible to form the product integral $\int_C(1+\alpha)$, the value (depending on the path of integration) an element of A. This integral maps paths C into A, preserving multiplication. Analytic conditions on α are derived in order that this map depend only on the homotopy class of the closed curve C. Thus, α is a representation of the homotopy group of M into A, if and only if $d\alpha+\alpha\wedge\alpha=0$, or equivalently, $\partial A_j/\partial x^i-\partial A_i/\partial x_j+A_iA_j-A_jA_i=0$. α is a trivial representation if and only if $\alpha=F^{-1}dF$, F an A-valued function (with inverse) on M. These are the analogues of exact and bounding differentials. In this way, representations of the homotopy group (into A) are obtained, and these groups—and homology groups over the integers—may be studied by means of differentials. Various examples are considered. (Received February 14, 1957.)

481. N. D. Kazarinoff: *D. K. Kazarinoff's inequality for tetrahedra.*

Let S be a tetrahedron and P be a point not exterior to S. Let the distances from P to the vertices and faces of S be denoted R_i and r_i, respectively. An analogue of the Erdös-Mordell inequality for triangles is established. THEOREM: For any tetrahedron whose circumcenter is not an exterior point, $(\sum R_i)/\sum r_i>2\cdot 2^{1/2}$, and $2\cdot 2^{1/2}$ is the greatest lower bound. Let the vertices of S be i, j, k, and l, (ij) be the length of the edge joining i and j, H_i be the length of the altitude through i, R be the radius of the circumsphere with center O, and r_i be the distance from P to the face opposite i. D. K. Kazarinoff proved the following lemma: $\sum R_i\geqq\sum[(ij)^2+(ik)^2+(il)^2]r_i/2RH_i$; equality holds if and only if P and O coincide. Using a method of successive minimization, one is able to show that the minimum value of the coefficient of an r in the last relation is $2\cdot 2^{1/2}$ in the case considered. The minimum is taken on only in the degenerate case where two vertices are opposite ends of a diagonal of a square, the other two vertices are at another corner, and P is the midpoint of the diagonal. (Received March 5, 1957.)

482. Wilhelm Klingenberg: *Isometry of Riemannian submanifolds. I.*

Let M^* be a Riemannian manifold (referred to as big manifold) of dimension d^* and let M be a submanifold of M^* (referred to as small manifold) of dimension d. Everything shall be of class C^∞. From the bundle $F^*(M^*)$ of orthonormal frames over M^* one has the induced bundle $F^*(M)$ of big frames over M. Furthermore one has, from the inclusion map $F^*(M)\to F^*(M^*)$, on $F^*(M)$ the induced forms ω_i and $\omega_{ij}=-\omega_{ji}$ $(i, j=1,\cdots,d^*)$. Here the ω_i determine the situation of the small tangent space at M in the big frames and the ω_{ij} determine the horizontal space in the tangent space at $F^*(M)$, i.e., they determine a connection in $F^*(M)$. Consider a second manifold $'M^*$ of dimension d^* and herein a submanifold $'M$ of dimension d. Again one has the induced bundle $'F^*('M)$ of big frames over $'M$ and the forms $'\omega_i$ and $'\omega_{ij}$ on the this bundle. By an isometry of $F^*(M)$ onto $'F^*('M)$ one means a 1:1 diffeomorphic bundle map Φ, which is compatible with the Riemannian structure of these bundles, i.e., $'\omega_i\circ d\Phi=\omega_i$ and $'\omega_{ij}\circ d\Phi=\omega_{ij}$. The object of the paper is to characterize the existence of such an isometry in a way that contains for the special case $M=M^*$ and $'M='M^*$ a recent theorem of Ambrose (Ann. of Math. vol. 64). (Received February 19, 1957.)

483*t*. Wilhelm Klingenberg: *Isometry of Riemannian submanifolds. II.*

(For notations see the preceding abstract.) M and $'M$ are assumed to be complete and simply connected. Fix a point m in M and a big frame at m, of which the first d vectors are in the small tangent space M_m of M at m. In the same way fix a point $'m$ in $'M$ and a big frame at $'m$. This determines a bundle map J of the trivial principal fibre bundle $M_m \times O(d^*)$ onto the bundle $'M'_{'m} \times O(d^*)$ ($O(d^*)$ is the structure group of $F^*(M)$). J induces a correspondence between the singly broken geodesics on M, emanating from m, and the singly broken geodesics on $'M$, emanating from $'m$, cf. Ambrose l.c. Using the parallel translation of the big frames along those singly broken geodesics by means of the big connection one gets a correspondence between the fibres of $F^*(M)$ and the fibres of $'F^*('M)$. If one restricts this correspondence to sufficient small neighborhoods of the endpoints of corresponding geodesic segments, emanating from m respectively from $'m$, then this correspondence becomes a 1:1 bundle map, to which we refer as local bundle maps. The main theorem states: If these local bundle maps are isometric, then there is an unique (global) isometry Φ from $F^*(M)$ onto $'F^*('M)$, which maps the fixed big frame at m into the fixed big frame at $'m$. Locally, Φ coincides with the given isometric local bundle maps. (Received February 19, 1957.)

484. T. S. Klotz: *On G. Bol's proof of Carathéodory's conjecture.*

Carathéodory conjectured that there exist at least two umbilic points on every closed surface of genus zero (with continuously turning tangent plane and continuous Gaussian curvature). This conjecture has been proved twice for the case of analytic surfaces, first by H. Hamburger (see Ann. of Math. vol. 41 (1940) pp. 63–86; Acta Math. vol. 73 (1941) pp. 174–332) and next by G. Bol. (see Math. Zeit. vol. 49 (1943–1944) pp. 389–410). There is a gap in Bol's paper and arguments have been found to fill the gap. The new arguments involve a careful application of lemmas already proved in Bol's paper. The corrected version of Bol's proof is still considerably shorter than Hamburger's. (Received February 19, 1957.)

485. Z. A. Melzak: *Convex hulls of a class of closed space curves.* Preliminary report.

Let C be a rectifiable closed curve or a rectifiable open arc in E_n and let $\overline{C}$ denote its convex hull. Let $L(C)$ and $V(\overline{C})$ denote, respectively, the length of C and the volume of $\overline{C}$. Let (c, n) (or (o, n)) stand for the problem of maximizing $V(\overline{C})$ subject to the condition $L(C) =$constant, for a closed curve C (or an open arc C). The problems $(c, 2n)$, $(o, 2n)$ and $(o, 2n+1)$ have been treated and solved under certain restrictions on C (Schoenberg, Acta Math., 1954; Egervary, 1949, Publ. Math. Debrecen). Here $(c, 3)$, posed by Bonnesen and Fenchel in 1934, is treated and solved under certain (rather stringent) conditions on C. By means of elementary symmetrization techniques it is shown that $(c, 3)$ is equivalent to an isoperimetric problem whose Euler-Lagrange equations are: $z' = xy$, $x'' = -xy^2$, $y'' = -yx^2$; here the independent variable is the arc-length and x, y, z are Cartesian coordinates of C. This work was supported by the Office of Naval Research. (Received February 14, 1957.)

486. D. J. Newman: *Length preserving transformations.*

An arbitrary mapping of the plane into itself is given. We prove: If this mapping preserves the distance 1, then it preserves all distances. This is a generalization of an old Putnam problem, viz.: If the mapping preserves all rational distances, then it preserves all distances. (Received January 14, 1957.)

487t. H. W. E. Schwerdtfeger: *Remarks on Moebius transformations.*

Let $Z=H(z)=(az+b)/(cz+d)$, $c\neq 0$. It is known that its fixed points and the poles α, $\beta(H(\alpha)=\infty$, $H(\infty)=\beta)$ are the pairs of opposite vertices of a parallelogram (cf. E. Jacobsthal, *Über die Klasseninvariante ähnlicher linearer Abbildungen* II, Kon. Norske Vid. Selskabs Forhandlinger vol. 26 (1953) pp. 10–15) which may degenerate into a segment. The straight lines through one of the fixed points, carrying a pair of sides of this "characteristic" parallelogram of H are mapped onto one another by H; this mapping is a perspectivity having the other fixed point as centre. Similar Moebius transformations H and SHS^{-1} possess similar characteristic parallelograms and conversely. Every transformation K similar to H can be represented in the form $K=THT^{-1}$ where $z^*=T(z)=uz+v$ is an integral Moebius transformation. This T is unique except when H is an involution in which case there are two different T. (Received February 8, 1957.)

Statistics and Probability

488t. T. E. Harris: *The random functions of cosmic-ray cascades.*

Consider the formulation for electron-photon cascades in, e.g., Jánossy's *Cosmic rays*, 2d ed., Oxford, 1950, pp. 206–208. Since Jánossy's $\mathfrak{H}(w, w')$ is a function of w'/w, we write $(w'/w)\mathfrak{H}(w, w')=K(w'/w)$. We use β rather than o for the collision-loss parameter. Let $N(y, t)$ be the number of electrons of energy $>y$ at thickness t; $f_i(s, y, t)=E(s^N)$ if the initial particle of energy 1 is of type $i=1$ (photon) or 2 (electron); electrons of zero energy are not counted. (1) If $\beta>0$, then $E\,N^2(0, t)<\infty$ (it was known that $E\,N(0, t)<\infty$), but with probability 1 $N(0, t)$ reaches ∞ in every t-interval. (2) If $\beta=0$, the total energy of all electrons converges in mean square and in probability to a constant as $t\to\infty$ (it was known that the expectation converges). (3) If $\beta=0$, the f_i are shown to have the necessary smoothness properties to satisfy (λ being a certain constant) $\partial f_1(s, y, t)/\partial t=-\lambda f_1(s, y, t)+\lambda\int_0^1 f_2(s, y/u, t)f_2(s, y/(1-u), t)\cdot uK(1/u)du$, $\partial f_2(s, y, t)/\partial t=\int_0^1\{f_1(s, y/u, t)f_2(s, y/(1-u), t)-f_2(s, y, t)\}\cdot K(u)du$. Although, starting with an electron, there is no first photon emission, it is also possible to get integral equations for the f_i by a modification of the "regeneration-point" method. (Received February 19, 1957.)

489. Bayard Rankin: *Computable probability spaces.*

Let $S=(a_1, a_2, \cdots)$ be uniformly dense in the unit interval I. Let X be a real valued Riemann integrable function on I with values $X(\alpha)$, $\alpha\in I$. Define $Ef(X)=\lim\,(1/N)\sum_1^N f[X(a_k)]$. For real continuous f, $0\leqq f\leqq 1$, with real domain $f(X)$ is Riemann and $Ef(X)$ equals the Riemann integral of $f(X)$ over I. X restricted to the space S is defined as a random variable with sample values $X(a_k)$ and mathematical expectation EX. The random events obtainable through X are defined as $f(X)$ with corresponding probabilities $Ef(X)$ for all f given above. In this context a modified weak law of large numbers holds for any strictly stationary and metrically transitive Riemann integrable sequence $X_1, X_2, \cdots$ that is defined on I and restricted to S. Namely, for any event $f(X)$ with $f(\beta)=0$, $Ef((1/N)\sum_1^N X_k)\to 0$ for β in a neighborhood of EX_k. S can be chosen so that each a_k is computable in the Turing sense and there exist X_k satisfying the above law all of whose sample values are computable. The same is true when "strictly stationary and metrically transitive" is strengthened to "statistically independent and identically distributed." (Received February 20, 1957.)

Topology

490. J. J. Andrews: *A characteristic of light open maps of locally Euclidean spaces into locally Euclidean spaces.*

A closed subset X of Euclidean n-space R^n is finitely nonseparating if and only if $R^n - X$ has no bounded components. A map f of R^n into R^m is pseudo-monotone if $f^{-1}(X)$ is finitely nonseparating for every finitely nonseparating set X. A map f of a locally Euclidean space X into a locally Euclidean space Y is locally pseudo-monotone if for every open set U in Y homeomorphic to $R^m(h\colon U \rightarrow R^m)$ and for every open set V in X homeomorphic to R^n $(g\colon R^n \rightarrow V)$ such that $f\colon V \rightarrow U$, then hfg is pseudo-monotone. Theorem: A light map of a locally Euclidean space into a locally Euclidean space is open if and only if f is locally pseudo-monotone. (Received March 7, 1957.)

491. T. R. Brahana: *Axioms for local homology.*

In this paper the axioms of Eilenberg and Steenrod, (Foundations of Alg. Top.), are adapted to fit local homology theory, and a uniqueness theorem for local homology on locally triangulable spaces is obtained. Local homology theory differs from homology theory in two main aspects; the local homology groups are invariants of local character, and homotopic maps do not, in general, induce the same homomorphisms of local homology groups. In outline, the development is as follows. Local categories and local homotopies are defined. The Axioms 1–5 of Eilenberg and Steenrod are used without change, 6, 7 modified slightly, and an Axiom 8 (Local) providing for local character of invariants added. A Local Simplicial Approximation Theorem is proved. In the development of homology of a simplicial pair, retraction to a unit interval replaces retraction to a point. Finally a Main Isomorphism Theorem and a Uniqueness Theorem analogous to that of Eilenberg and Steenrod, (l.c., p. 100), are proved. It is a consequence of this theorem that local homology theories such as those of van Kampen, (Thesis, Den Haag), Seifert and Threlfall, (Lehr. der Top., p. 121), Alexandroff, (Annals of Math. vol. 36, p. 1), Wilder, (Top. of Manifolds, p. 191), White, (Canadian Journal of Mathematics vol. 4, p. 329), and others, lead to the same invariants in locally triangulable spaces. (Received February 11, 1957.)

492. M. L. Curtis and M. K. Fort, Jr. (p): *Homotopy groups of one-dimensional spaces.*

Let X be a locally connected continuum whose one-dimensional integral singular homology group is a torsion group, and let S be a one-dimensional separable metric space. It is shown that any map $f\colon X \rightarrow S$ is homotopic to a constant map. From this result it follows immediately that $\pi_k(S) = 0$ for all $k > 1$. If S is also arcwise connected and $\pi_1(S)$ is a free finitely-generated group, then all singular homology groups $H_k(S, Z) = 0$, $k > 1$. (Received February 13, 1957.)

493. M. L. Curtis (p) and M. K. Fort, Jr.: *Certain subgroups of the homotopy groups.*

If $F_n(X, x_0)$ is the set of all maps of (S^n, y_0) into (X, x_0), then there is a natural map $\phi\colon F_n(X, x_0) \rightarrow \pi_n(X, x_0)$. A subset S of $F_n(X, x_0)$ is called a *G-class* if $\phi(S)$ is a subgroup of $\pi_n(X, x_0)$. A map $f\colon A \rightarrow B$ is *k-light* if $\dim f^{-1}(b) \leqq k$ except for a finite number of b in B. A map $f\colon A \rightarrow B$ is *k-monotone* if $H_k(f^{-1}(b))$ is finitely generated for each b in B. The set D of k-light elements of $F_n(X, x_0)$ is a G-class and the subgroup

$\phi(D)$ is denoted by $D_n^k(X, x_0)$. The set M of k-monotone elements of $F_n(X, x_0)$ is a G-class (with, for example, Cech homology over a field) and the subgroup $\phi(M)$ is denoted by $M_n^k(X, x_0)$. There are simple examples in which these subgroups are proper, and, of course, these subgroups are not homotopy-type invariants. If X is a finite polyhedron, then $M_n^k(X, x_0)=\pi_n(X, x_0)$ for all n, k. If dim $X \leqq m$, then $D_n^k(X, x_0)=0$ for $n>m+k$. If X is an m-manifold, then $D_n^k(X, x_0)=\pi_n(X, x_0)$ for $n \leqq m+k$. (Received February 13, 1957.)

494*t*. I. S. Gál: *On completely* (m, n)*-compact spaces.*

A topological space X is called (m, n)-compact if for every open cover $\{0_i\}$ $(i \in I)$ of X whose cardinality card $I \leqq n$ one can select a subcover $\{0_{i_j}\}$ $(j \in J)$ such that card $J \leqq m$. A space X is called completely (m, n)-compact if every subspace of X is (m, n)-compact. Theorem: *X is completely (m, n)-compact if and only if any one of the following statements holds in X:* (i) *Every subset $S \subseteq X$ satisfying $m \leqq$* card *$S \leqq n$ contains an m-accumulation point.* (ii) *If $S \subseteq X$ and* card *$S \leqq n$ and if there is no nonvoid subset in S which is m-dense in itself then* card *$S \leqq m$.* (iii) *If $\{0_i\}$ $(i \in I)$ is a well ordered increasing family of distinct open sets in X and if* card *$I \leqq n$ then* card *$I \leqq m$.* If the Cartesian product of infinitely many factors is completely (m, n)-compact then every subproduct of finitely many factors is completely (m, n)-compact. We have the following converse: Theorem: *Suppose that every product space of finitely many factors selected from a family $\{X_i\}$ $(i \in I)$ of topological spaces is completely (m, ∞)-compact for some m satisfying* card *$I \leqq m$. Then the topological product $X=\prod_{i \in I} X_i$ is also (m, ∞)-compact.* (Received February 20, 1957.)

495*t*. Bert Mendelson: *Homology of symmetric products of spheres.* Preliminary report.

Let ${}_kS^n$ denote the k-fold symmetric product of an n-sphere. ${}_kS^n$ is given the structure of a CW complex such that the sequences of integers $(i_1, \cdots, i_s)$ with $0 \leqq s \leqq k-1$ and $2 \leqq i_j \leqq n$ are in one-to-one correspondence with the $(n+i_1+\cdots+i_s)$-cells, denoted by $[n; i_1, \cdots, i_s]$ of ${}_kS^n$. The homology boundary of $[n; i_1, \cdots, i_s]$ is computed. If ${}_kS^n$ is embedded in ${}_{k+1}S^n$, the chain complex based on the cells $[n; i_1, \cdots, i_s]$, $0 \leqq s \leqq k$, is the direct sum of the chain complex of ${}_kS^n$ and the chain complex based on the cells $[n; i_1, \cdots, i_k]$ of $({}_{k+1}S^n, {}_kS^n)$. Thus the integral homology of ${}_{k+1}S^n$ is given by $H({}_{k+1}S^n)=H({}_kS^n)+H({}_{k+1}S^n, {}_kS^n)$ (direct), a special case of a result of Steenrod's. Let $J(k+1; n; 2r)$, $r=1, 2, \cdots, [n/2]$ be the chain group generated by the cells $[n; i_1, \cdots, i_k]$ in which either $2r$ or $2r+1$ occurs among the set $\{i_1, \cdots, i_k\}$, but no smaller integer occurs. It is shown that $J(k+1; n; 2r)$ is a subcomplex and that the chain complex of $({}_{k+1}S^n, {}_kS^n)$ is the direct sum $\sum J(k+1; n; 2r)$, and that furthermore $J(k+1; n; 2r)$ is chain isomorphic to $J(k+1; n+2; 2(r+1))$, by an isomorphism raising dimension by $2(k+1)$. Thus, r running from 0 to $[n/2]-1$, $H({}_{k+1}S^n, {}_kS^n) \approx \sum H(J(k+1; n-2r; 2))$. An inductive procedure is devised for computing $H(J(4; n; 2))$ for all n, thus determining $H({}_4S^n, {}_3S^n)$. Also generating cycles are explicitly arrayed leading to calculations of reduced products. (Received December 13, 1956.)

496. E. A. Michael: *Continuous extensions and selections for paraconvex sets.*

Let E be a Banach space, with distance ρ. A set $A \subset E$ is called (n-point) paraconvex if there exists an $\alpha < 1/2$ such that, for any $S \subset A$ (containing $\leqq n$ points) and any $x \in \text{conv } S$, $\rho(x, A) \leqq \alpha(\text{diam } S)$. Examples: Every convex set is paraconvex, and so is a proper closed subarc of a semi-circle or the letters $\boldsymbol{X}$, $\boldsymbol{Y}$, or $\boldsymbol{Z}$ in the euclidean plane; a regular pentagon is 2-point paraconvex but not 3-point paraconvex. Theorem: If $\mathcal{S}$ is a family of nonempty, closed (n-point) paraconvex sets in E for which the same α works, and if X is a paracompact space (of Lebesgue dimension $\leqq n-1$), then every lower semi-continuous $\phi: X \rightarrow \mathcal{S}$ admits a selection. (This generalizes Theorem 3.2″ of the author's paper [Ann. of Math. vol. 63 (1956) pp. 361–382], where terminology is explained.) Corollary: If $A \subset E$ is closed and (n-point) paraconvex, then A is an extension space for all metric, and hence all collectionwise normal, spaces (of Lebesgue dimension $\leqq n-1$). (Received February 6, 1957.)

497. J. P. Roth: *On the existence of a continuum of critical points.*

Let f be a map of a Hausdorff space S into the reals such that $S_c = \{x | f(x) \leqq c\}$ is compact. Let u and v be nonzero elements of Čech homology groups $H_r(S)$ and $H_{r+s}(S)$, $s>0$, with the same critical value c [Bull. Amer. Math. Soc. Abstract 60-2-331]. Let u_* be a cocycle dual to u under an orthogonal pairing [Wilder, *Topology of manifolds*, American Mathematical Society Colloquium Publications, no. 32, 1949.] If the cap product $u_* \cap v$ is not zero, then there is an s-dimensional set of critical points at level c. Corollaries are generalizations of the Lyusternik-Schnirelmann theorems of the existence of critical sets of differentiable maps of projective n-space and the n-torus. The proofs do not utilize the notion of category. (Received February 19, 1957.)

498. F. J. Wagner: *Almost homogeneous spaces.*

Let X be a topological space, and G a topological group such that, corresponding to every $g \in G$, there exists a homeomorphism (also denoted by g) satisfying the following conditions: (1) if $g, h \in G$, then $g(h(x)) = (gh)(x)$ for every $x \in X$; (2) for every $g \neq e$ of G, there exists an $x \in X$ such that $g(x) \neq x$; (3) for every $x \in X$, the set $\{g(x) | g \in G\}$ is dense in X. Then X is called an almost homogeneous space (relative to G). Let $X = \bigcup C(\alpha)$ be a decomposition of X into its components. For every $g \in G$, $g(C(\alpha)) = C(g(\alpha))$ is a component of X, for every α. $H = \{g \in G | g(\alpha) = \alpha \text{ for every } \alpha\}$ is a closed normal subgroup of G. G/H is a group of permutations of the α's. $N(\alpha) = \{g \in H | g(x) = x \text{ for every } x \in C(\alpha)\}$ is a closed normal subgroup of H, and H is isomorphic to a subgroup of the direct product of the groups $H(\alpha) = H/N(\alpha)$. If for every α, α' there is a $g \in G$ such that $g(\alpha) = \alpha'$ (e.g. if X is locally connected), then all the components $C(\alpha)$ are of course homeomorphic and all the groups $H(\alpha)$ are isomorphic. (Received February 21, 1957.)

499. G. W. Whitehead: *Group extensions defined by cohomology classes.*

Let X be a free graded chain complex. Then each $u \in H^q(X; G)$ defines canonically a homomorphism $u': H_q(X) \rightarrow G$, as well as an element $u'' \in \text{Ext}\,(H_{q-1}(X), \text{Cok } u')$. If (X, A) is a pair, $i_q: H_q(A) \rightarrow H_q(X)$ is the injection, and $u \in H^q(X, H_q(X))$ is an element such that $u' = \text{identity}$, then the extension $0 \rightarrow \text{Cok } i_q \rightarrow H_q(X, A) \rightarrow \text{Ker } i_{q-1} \rightarrow 0$ is determined by the injection into Ext (Ker i_{q-1}, Cok i_q) of the element $(i_q{}^*u)''$. Let X be an $(n-1)$-connected space, let $u \in H^n(X, \pi_n(X))$ be the first obstruction to contracting X to a point, and let $v \in H^{n+2}(X; \Gamma_{n+1}(X))$ be the Pontryagin (Steenrod) square of u if $n=2$ $(n>2)$; then, in the exact sequence $H_{n+2}(X) \rightarrow^b \Gamma_{n+1}(X) \rightarrow \pi_{n+1}(X)$

$\rightarrow H_{n+1}(X) \rightarrow 0$ of J. H. C. Whitehead, we have $v' = b$, while the extension $0 \rightarrow \mathrm{Cok}\, b \rightarrow \pi_{n+1}(X) \rightarrow H_{n+1}(X) \rightarrow 0$ is defined by v''. Let X be a space with just two nonvanishing homotopy groups; then a formal description of the homology groups of X in the stable range can be given in terms of u', u'', where u ranges over a set of cohomology classes which can be expressed by the action of iterated Steenrod operations on the Postnokov invariant of X. (Received March 28, 1957.)

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