THE FEBRUARY MEETING IN NEW YORK

The five hundred twenty-second meeting of the American Mathematical Society was held at Hunter College in New York City on Saturday, February 25, 1956. The meeting was attended by about 325 persons including 286 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings Professor Everett Pitcher of Lehigh University delivered an address entitled *Inequalities of critical point theory* at a general session presided over by Professor R. E. Langer. Sessions for contributed papers were held in the morning and afternoon, presided over by Professors J. H. Barrett, A. A. Bennett, R. H. Bing, G. K. Kalisch, D. E. Kibbey, and C. H. W. Sedgewick.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, the paper was presented by that author whose name is followed by "(p)". Dr. Banaschewski was introduced by Professor D. B Sumner, Drs. Bremermann and Huckemann and Miss Weiss by Professor R. D. Schafer, Mr. Friedberg by Dr. Hartley Rogers, Jr., Professor Kalish and Dr. Montague by Professor Alfred Horn, Professor Kreyszig and Dr. Nitsche by Professor Stefan Bergman, Dr. Laugwitz by Professor E. R. Lorch, and Mr. Robinson by Professor R. F. Rinehart.

ALGEBRA AND THEORY OF NUMBERS

283*t*. R. M. Baer: *Certain homomorphisms of partially ordered sets.*

It is well known that for an arbitrary partially ordered set X there exists a chain C and a homomorphism σ which carries X onto C. [Tarski, Kuratowski, Szpilrajn.] The authors calls a homomorphism of X onto C *thorough* with respect to a collection of chains (D_α) in X if the homomorphism maps every chain D_α *onto* C. In terms of a notion of *gap* in a partially ordered set X, those $(X, (D_\alpha))$ are characterized for which, given C, there exists a homomorphism which is thorough with respect to a nonempty collection of maximal chains in X and which maps X onto C. (Received January 13, 1956.)

284. D. W. Blackett: *Simple near-rings of differentiable transformations.*

This paper examines simple near-rings of transformations of a finite dimensional real vector space V into itself. The particular transformations considered have the origin θ as a fixed point and are continuously differentiable at θ. A simple near-ring N of such transformations is faithfully represented as a ring of linear transformations by restriction to the invariant subspace of V generated by the images of a fixed vector not annihilated by N. This statement must be modified by the restriction that at least one transformation in N has some nonvanishing first partial derivative at θ. One consequence is that the ring of linear transformations of V is a maximal simple near-ring in the near-ring of all transformations of V with θ as a fixed point and a point of continuous differentiability. Another corollary is that the simple near-rings of entire functions over the complex plane which vanish at 0 are the rings of transformations $z \to bz$ where b ranges over a subfield of the complex numbers. (Received January 3, 1956.)

285. J. R. Büchi: *On the representation of Boolean rings by sets.* Preliminary report.

$\langle B, +, 0\rangle$ is a *Boolean group* (B.G.) if it is a group satisfying the law $x+x=0$, it is a *Boolean group of sets* if B is a class of sets closed under the operation $\oplus$ of symmetric sum and '+' denotes $\oplus$. Let $x \leftrightarrow x^*$ be a one-to-one mapping of a $+-$basis S into sets x^* such that, (1) $S^* = [x^* | x \in S]$ is $+-$independent. (2) $h(x_1 + \cdots + x_n) = x_1^* \oplus \cdots \oplus x_n^*$ then defines an isomorphism h of the B.G. onto the B.G. of sets, $\oplus$-generated by S^*. By Zorn's lemma there exist $+-$ bases for every Boolean group, furthermore $x^* = [x]$ satisfies (1). Therefore, (I) *every B.G. is isomorphic to a B.G. of sets* (*of all finite subsets of a set*). To represent Boolean rings $\langle B, +, \cdot, 0\rangle$ an additional condition is required, (3) $x, y, u_1, \cdots, u_n \in S$, $x \cdot y = u_1 + \cdots + u_n \to x^* \odot y^* = u_1^* \oplus \cdots \oplus u_n^*$. (II) *If S is a $+-$basis of a B.R. and $x \leftrightarrow x^*$ satisfies* (1) *and* (3), *then* (2) *defines a representation of the B.R. by a B.R. of sets.* Combined with the following fact this yields rather economical representations for a wide class of B.R.'s, including such which have a chain-basis. (III) *If S is a $+-$ basis of a B.R., if $S \cup [0]$ is closed under $\cdot$, and if $D \subset S$ is "dense" in S, then $x^* = [u | u \in D, u \subseteq x]$ satisfies* (1) *and* (3). (Received January 13, 1956.)

286*t*. Harvey Cohn: *Equivalence of theta reciprocity and Gaussian sum reciprocity.* I.

The method of Cauchy and Hecke establishes Gaussian sum reciprocity as a consequence of theta reciprocity through the asymptotic value of both members of the latter relation. To go in the opposite direction we note a complete expansion is equivalent to the following identity: $\exp[\pi i S(\text{sgn } \omega)/4] g^{-n} a^{-1} \sum \exp[-\pi i S(\omega \lambda^{*2})] \cos 2\pi S \cdot [\mu \lambda^*/g] = N(\omega)^{1/2} G^{-n} \sum \exp[\pi i S(\mu^{*2}/\omega) + \pi i S(\omega\lambda^2)/G^2] \cos 2\pi S[\lambda\mu^*/G]$. Here $\mathfrak{a}, \mathfrak{A}$ are reciprocal algebraic modules, ($\mathfrak{a}$ of determinant a) and ω is an algebraic number of the corresponding field with S the trace operator; g, G are integers depending only on $\mathfrak{a}$, $\mathfrak{A}$, ω; $\lambda^* \in \mathfrak{a}$, $\mu^* \in \mathfrak{A}$ are summed over g^n, G^n residue classes mod $g\mathfrak{a}$ and mod $g\mathfrak{A}$; and $\lambda \in \mathfrak{a}$, $\mu \in \mathfrak{A}$ are connected by $\mu/g = \pm \omega\lambda/G$ a vacuous sum being represented by zero. The identities hold for such λ, μ provided they hold for only a finite number of residue classes. This is another way of saying $\mathfrak{a}$, $\mathfrak{A}$ are reciprocal modules if the basis relations satisfy the reciprocal relation modulo M for a finite integer M. (Received January 11, 1956.)

287. M. P. Epstein: *Derivations of differential fields.* Preliminary report.

Let F be an ordinary differential field of characteristic zero with field of constants C, Ω a universal extension of F with field of constants C^*. Denote by δ the differentiation operator in Ω. Let G be a differentially algebraic extension of F of transcendence degree n. Define a (differential) derivation of G/F to be a map D: $G \to \Omega$ satisfying: (1) for $\alpha, \beta \in G$, $D(\alpha+\beta) = D\alpha + D\beta$, $D(\alpha\beta) = \alpha D\beta + \beta D\alpha$; (2) for $a \in F$, $Da = 0$; (3) $D\delta = \delta D$. Call D restricted if $D(G) \leqq G$. Then the set of derivations of G/F forms a

Lie algebra over C^* of dimension exactly n. However there may be no nontrivial restricted derivations of G/F. But when G is a Picard-Vessiot extension of F with the same field of constants C then there are n linearly independent restricted derivations of G/F so the set of all restricted derivations of G/F is a Lie algebra $\mathfrak{g}$ over C. Kolchin showed that the group of automorphisms of G/F may be identified with an algebraic matric group $\mathfrak{G}$ over C. When this is done, $\mathfrak{g}$ may be naturally identified with the Lie algebra of $\mathfrak{G}$. When G is strongly normal over F again there are n linearly independent derivations of G/F. (Received January 11, 1956.)

288*t*. G. D. Findlay and Joachim Lambek: *On rings with projective ideals.*

Let R be a ring having a unit element and assume that every left ideal of R is projective. Then every submodule of any projective unitary left R-module is a direct sum of modules isomorphic to left ideals of R, and is therefore also projective. This generalizes a theorem by Everett (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 312–316). If M and N are any two left R-modules, then the canonical homomorphism from the tensor product $M^* \cdot N$ into Hom (M, N) is faithful. This generalizes a theorem by Whitney (Duke Math. J. vol. 4 (1938) pp. 495–528). (Received January 16, 1956.)

289. H. K. Flesch: *Finite elementary nilpotent groups of class* 2.

Let EN2-group denote a group in which every element except identity is of the same odd prime order p and the commutator group and center coincide. If G is the EN2-group with minimal basis of length k whose commutator group C has the full order $p^{\binom{k}{2}}$, all EN2-groups are contained in the classes $G(p, k, d)$ of factor groups G/S with S of order p^d in C. The effect on $c \in G$ of an automorphism α of G is described by a congruence $\tilde{c} \rightarrow A\tilde{c}A'$, where A is the matrix of the transformation induced by α on the vectorspace G/C, and $\tilde{c}$ a skew-symmetric matrix isomorphically corresponding to c. Since every automorphism of G/C can be lifted to G, it follows that $G/S \simeq G/T$ if and only if there exists A such that $A\tilde{S}A' = \tilde{T}$. Hence classification of $G(p, k, d)$ reduces to that of the homogeneous d-parameter families of skew k by k matrices over $GF(p)$ under congruence and linear parameter transformation. This is trivial for $d = 1$ and manageable for $d = 2$ by reference to invariant factor theory. Enumerations verify in particular all previous results for $d \leqq 2$. An example shows that EN2-groups exist for d up to $\binom{k}{2} - 1$ when k is even. (Received January 10, 1956.)

290. I. N. Herstein: *Lie and Jordan systems in simple rings with involution.*

Let A be a simple ring with an involution *, and let S be the set of self-adjoint elements of A and K the set of skew elements of A. S is a Jordan ring under the product $a \circ b = ab + ba$ and K is a Lie ring under $[a, b] = ab - ba$. In this paper it is shown that if the characteristic of A is not 2 then S is a simple Jordan ring; also, if in addition A is more than 16 dimensional over its center, the only Lie ideals of K are either in the center of A or contain $[K, K]$. It is also shown that if A is more than 4 dimensional over its center S generates A; also, in this situation the subring generated by K is A. (Received January 11, 1956.)

291. D. R. Hughes: *A class of non-Desarguesian projective planes.*

In 1907 Veblen and Wedderburn gave an example of a projective plane of order nine, none of whose planar ternary rings are Veblen-Wedderburn systems (with either distributive law); this plane is sometimes known as the "Carmichael plane." This plane is shown to be self-dual (thus establishing that there are exactly four planes of order nine known at the present time) and to possess at least 78 distinct Fano subplanes. Furthermore, the Veblen and Wedderburn technique is generalized to give a non-Desarguesian plane π of order p^{2n}, for any odd prime p and any positive integer n, with the following properties: none of the planar ternary rings for π are Veblen-Wedderburn systems, but (at least) one of the rings has the elementary abelian group of order p^{2n} as its additive loop. The question of self-duality is still open for every case excepting $p^{2n}=9$. (Received December 14, 1955.)

292t. S. A. Jennings and Rimhak Ree: *On a family of Lie algebras of characteristic p.*

Let A be the group algebra over an algebraically closed field Φ of characteristic p of an abelian p-group of order p^n and of the type $(p, p, \cdots, p)$, $D_0, \cdots, D_m$ derivations of A satisfying the conditions (i) $D_i \circ D_j = 0$ for all i, j; (ii) $\sum f_i D_i = 0$, where $f_i \in A$, implies $f_i = 0$ for all i; (iii) If $f \in A$ is such that $D_i f = \lambda_i f$, where $\lambda_i \in \Phi$, for all i, then $f=0$ or f is a unit in A; (iv) $D_i f = 0$ for all i implies $f \in \Phi$. Let $a_0, \cdots, a_m \in A$ be such that $D_i a_j = D_j a_i$ for all i and j. Then the set L of all derivations of A of the form $f_0 D_0 + \cdots + f_m D_m$, where $f_i \in A$ satisfy $\sum (D_i f_i - a_i f_i) = 0$, forms a subalgebra of the derivation algebra of A. It is shown that dim L is either mp^n+1 or mp^n. For the former case L is said to belong to the family F_{I}, and for the latter case, the family F_{II}. It is shown that algebras in a certain subfamily of F_{II} are simple, that algebras in a subfamily of F_{I} have simple first derived algebras of dimension $m(p^n-1)$, where m, n may be arbitrary integers such $1 \leqq m < n$, and that if $m=1$ then certain algebras in F_{I} have simple second derived algebras of dimension p^n-2, where $n>1$, $p>2$. The dimension $m(p^n-1)$ is in general new. The simple algebras constructed by M. S. Frank [Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954)] are contained in F_{I}. However, the connection between the algebras in F_{I} and those constructed by A. A. Albert and Mrs. M. S. Frank [Bull. Amer. Math. Soc. Abstract 61-4-530] is undecided. (Received January 13, 1956.)

293t. Naoki Kimura: *Decomposition of homogeneous semigroups.*

A semigroup S is called homogeneous if for any two elements $x, y \in S$ there exists an automorphism of S sending x to y. A semigroup S is called idempotency-simple or commutativity-simple, if any homomorphic image of S which is idempotent or commutative is reduced into one element semigroup respectively. Under these definitions any homogeneous semigroup S is decomposed in the following manner: $S \cong A \times B \times C \times D \times E$, where A is a left singular semigroup, i.e., for any elements $x, y \in A$, $xy = x$, B is a right singular semigroup, C is a homogeneous semilattice, D is an idempotency-simple homogeneous commutative semigroup and E is both an idempotency-simple and a commutativity-simple homogeneous semigroup. (Received January 13, 1956.)

294t. Joseph Landin and Irving Reiner: *Automorphisms of the Gaussian unimodular group.*

Let G_n denote the group of $n \times n$ unimodular matrices over the ring of Gaussian integers. It is proved that the automorphism group $\mathfrak{A}_n$ of G_n is generated by the following automorphisms: (i) inner, (ii) $X \rightarrow X'^{-1}$, (iii) $X \rightarrow \overline{X}$ (conjugate), (iv) $X \rightarrow (\det X)^k X$ where $k=1$ for even n, and $k=2$ for odd n, and (v) $(P, S, T) \rightarrow (P, -S, -T)$ for $n=2$ only, where $(x, y)P = (ix, y)$, $(x, y)S = (-y, x)$, $(x, y)T = (x, x+y)$. For $n=2$, the automorphism (ii) is superfluous. The proofs given depend upon canonical forms for involutions in G_n under conjugacy in G_n, a method due to C. E. Rickart (Amer. J. Math. vol. 72 (1950) pp. 451–464) for characterizing the $(1, n-1)$ involutions, and the use of maximal sets of involutions (I. Reiner, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 459–476) to distinguish between different kinds of $(1, n-1)$ involutions. (Received January 11, 1956.)

295*t*. M. D. Marcus: *An extension of the Minkowski determinant inequality.*

Theorem 1: Assume (i) A and B are n-square non-negative Hermitian matrices with complex entries; (ii) the eigenvalues of A, B, and $A+B$ are $0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_n$, $0 \leqq \mu_1 \leqq \cdots \leqq \mu_n$, $0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_n$ respectively; (iii) $E_r(a_1, \cdots, a_k)$ denotes the rth elementary symmetric function of the numbers $a_1, \cdots, a_k$. Then for $1 \leqq r \leqq k \leqq n$, $E_r^{1/r}(\lambda_1, \cdots, \lambda_k) \geqq E_r^{1/r}(\alpha_1, \cdots, \alpha_k) + E_r^{1/r}(\mu_1, \cdots, \mu_k)$. Theorem 2: Assume (i) and (ii). Let $p(A) = x^n + \sum_{j=1}^n p_j(A)x^{n-j}$ denote the characteristic polynomial of A. Then $|p_j(A+B)|^{1/j} \geqq |p_j(A)|^{1/j} + |p_j(B)|^{1/j}$. The case $j=n$ is the classical Minkowski determinant inequality. (Received December 19, 1955.)

296*t*. M. D. Marcus: *Eigenvalue inequalities for finite matrices.* I.

Let A and B be n-square Hermitian matrices with complex entries. Let λ_j, α_j, μ_j, ω_j, $j=1, \cdots, n$, be the eigenvalues of $C=A+iB$, A, B and $(C^*C)^{1/2}$ respectively, so arranged that $|\lambda_j| \leqq |\lambda_{j+1}|$, $\alpha_j \leqq \alpha_{j+1}$, $\mu_j \leqq \mu_{j+1}$, $\omega_j \leqq \omega_{j+1}$. Let $E_r(a_1, \cdots, a_k)$ be the rth elementary symmetric function of the a_j and let $p(A) = x^n + \sum_{j=1}^n p_j(A)x^{n-j}$ be the characteristic polynomial of A. Results: $0 \leqq \alpha_1$ and $1 \leqq r \leqq k \leqq n$ imply (i) $\prod_{j=1}^k |\lambda_j| \geqq \prod_{j=1}^k \alpha_j$; (ii) $|E_r(\lambda_1, \cdots, \lambda_n)| \geqq C_{n,r} \prod_{j=1}^r \alpha_j$; (iii) $E_r(\omega_1, \cdots, \omega_k) \geqq E_r(\alpha_1, \cdots, \alpha_k)$; (iv) $|p_r((C^*C)^{1/2})| \geqq |p_r(A)|$; (v) if A is real symmetric and B is real antisymmetric then $|p_r(A+B)| \geqq |p_r(A)|$. In case $r=k=n$, (i)–(v) specialize to a result of Ostrowski and Taussky (Neder. Akad. Wetensch. A. vol. 54 (1951) p. 383). If $0 \leqq \rho \leqq 1$ and $\sigma_j \leqq \sigma_{j+1}$ are the eigenvalues of $\rho A + (1-\rho)B$ and $0 \leqq \alpha_1$, $0 \leqq \mu_1$ then (vi) $E_r^{1/r}(\sigma_1, \cdots, \sigma_k) \geqq \rho E_r^{1/r}(\alpha_1, \cdots, \alpha_k) + (1-\rho)E_r^{1/r}(\mu_1, \cdots, \mu_k) \geqq E_r^{\rho/r}(\alpha_1, \cdots, \alpha_k)E_r^{(1-\rho)/r}(\mu_1, \cdots, \mu_k)$; (vii) $|p_r(\rho A + (1-\rho)B)| \geqq |p_r(A)|^\rho |p_r(B)|^{1-\rho}$. (vi) and (vii) generalize a result of Fan (Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) p. 31). (Received December 19, 1955.)

297*t*. M. D. Marcus and B. N. Moyls: *Eigenvalue inequalities for finite matrices.* II.

Let $0 \leqq \lambda_i \leqq \lambda_{i+1}$; $0 \leqq a_i \leqq a_{i+1}$ for $i=1, \cdots, k-1$, and assume $\sum_{i=1}^j (a_i - \lambda_i) \geqq 0$ for $j=1, \cdots, k$. (i) If f is a concave monotone nondecreasing function defined on $[\lambda_1, \max(\lambda_k, a_k)]$ then $\sum_{i=1}^k (f(a_i) - f(\lambda_i)) \geqq 0$. Let A be a positive definite n-square Hermitian matrix with eigenvalues $0 < \alpha_1 \leqq \cdots \leqq \alpha_n$. Let $E_r(a_1, \cdots, a_k)$ denote the rth elementary symmetric function of the a's for $1 \leqq r \leqq k \leqq n$. Set $\phi = E_r(f((Ax_1, x_1)), \cdots, f((Ax_k, x_k)))$ where $x_i, \cdots, x_k$ are an orthonormal (o.n.) set. Then if the function f in (i) satisfies $f(xy) = f(x)f(y)$ and is monotone increasing it follows that (ii) $\min \phi = E_r(\alpha_1, \cdots, \alpha_k)$, $\max \phi = C_{k,r}(f(\sum_{j=1}^k \alpha_{n-j+1}/k))^r$ where $x_1, \cdots, x_k$ run over all sets of k o.n. vectors. (ii) is applied to obtain inequalities connecting the eigenvalues of $(A+B)^{1/2}$, $A^{1/2}$ and $B^{1/2}$. (Received December 28, 1956.)

298*t*. M. D. Marcus and B. N. Moyls: *Eigenvalue inequalities for finite matrices.* III.

For notation see above abstract. It is known that minimizing and maximizing sets of k o.n. vectors for ϕ span an invariant subspace under A when A is positive definite (M. Marcus and J. L. McGregor *Extremal properties of Hermitian matrices*—to appear). It is shown here that for $f(x)=x$, $r=k$ (i.e. the product of k forms) the assumption on A need only be Hermitian nonsingular, to conclude the invariance. From this it follows that for $r=k=2$, max ϕ and min ϕ are max $((\alpha_i+\alpha_j)/2)^2$, min $\alpha_i\alpha_j$ respectively taken over all $i<j$ where $\alpha_i \leqq \alpha_{i+1}$ are the eigenvalues of A. For $r=2$, $k=n$ and A simply Hermitian it is also proved that min $\phi=p_2(A)$, the 2nd coefficient in the characteristic polynomial of A. (Received January 3, 1956.)

299. R. F. Rinehart: *Generalization of the concept of derivative of a matric function.*

Let $f(z)$ be a function of a complex variable and A a square matrix over the complex field. Let $f(A)$ be defined in the sense of Giorgi or any of the equivalent senses. In an earlier paper a definition of the derivative of the matrix function $f(Z)$ at $Z=A$ was given, in which the "incremental matrix" H was required to be commutative with A. In the present paper the following extension of that definition is given which removes that restriction: If $f(A+H)-f(A)$ is expressible, for $|h_{rs}|$ sufficiently small, in the form $\sum_{i=1}^{m} P_iHQ_i$, and if $\lim_{H\to 0} \sum_{i=1}^{m} P_iQ_i$ exists uniquely for any manner of approach of H to zero (i.e. $h_{rs}\to 0$), then this limit is called the derivative of $f(Z)$ at $Z=A$. It is shown that a necessary and sufficient condition that this derivative exist, is that the scalar function $f(Z)$ be analytic at each of the characteristic roots of A. The result $f'(A)=g(A)$, where $g(Z)=f'(Z)$ remains valid for this broader definition. (Received January 3, 1956.)

300. D. W. Robinson: *Functions of commutable linear transformations.*

A study is made of a defined composition, in a finite linear space over the field of complex numbers, between functions of several complex variables and sets of pairwise commutable linear transformations. The functions considered are not restricted to be simply power series, nor are the transformations required to be normal, as in previous studies. Equivalent forms of the composition are obtained, and some algebraic and topological properties are investigated. The results reveal the composition to be in both form and properties an extension of the previously studied case of a single variable. This is accomplished primarily by the formulation of the concept of projections of a set of pairwise commutative linear transformations, corresponding to prescribed eigenvalue arrangements. The eigenvalues of the defined linear transformations are also exhibited, generalizing and illuminating a result of Frobenius for rational functions. Finally, an extension is made of the class of admissible functions by means of the notion weak convergence. (Received January 6, 1956.)

301*t*. Robert Steinberg: *Prime power representations of some classical finite groups.*

There are five well-known two-parameter families of simple finite groups: the unimodular projective, the symplectic, the unitary, and the first and second orthogonal groups, acting on a vector space of a finite number of elements. The following is proved: Let G be one of the groups above. Let p be the characteristic of the base field, let d be the order of a p-Sylow subgroup P of G, and let m be the index of the normalizer of P in G. Let Σ be any vector space of dimension d over a field of characteristic 0 or prime to m. Then G has an irreducible representation of degree d with Σ as the representation space. The method is constructive and yields the representation space and the representing matrices explicitly. (Received January 10, 1956.)

302. Harold Widom: *Approximately finite algebras.*

Let $\mathfrak{M}$ be a factor of type II_1 on a (not necessarily separable) Hilbert space $\mathfrak{H}$. We introduce three notions of approximate finiteness for $\mathfrak{M}$. (A1) Given $A_1, \cdots, A_n \in \mathfrak{M}$ and $\epsilon > 0$ there exists a subfactor $\mathfrak{N}$ of $\mathfrak{M}$ of type I containing elements $B_1, \cdots, B_n$ such that $[[A_i - B_i]] < \epsilon$ $(i = 1, \cdots, n)$. (A2) If $\mathfrak{U}$ is a collection of subfactors of $\mathfrak{M}$ with the properties (a) if $\mathfrak{N}_\lambda$ is an increasing chain of members of $\mathfrak{U}$ then $R(\mathfrak{N}_\lambda) \in \mathfrak{U}$, and (b) $\mathfrak{U}$ contains the subfactors of type I, then $\mathfrak{M} \in \mathfrak{U}$. (B) There exist mutually commuting subfactors $\mathfrak{N}_\lambda$ of $\mathfrak{M}$ of type I such that $R(\mathfrak{N}_\lambda) = \mathfrak{M}$. In case $\mathfrak{H}$ is separable, (A1)–(B) are equivalent to the usual notion of approximate finiteness. In general (A1) and (A2) are equivalent, both being weaker than (B). Let $\chi(\mathfrak{M})$ be the density character of $\mathfrak{M}$ relative to the metric $[[\quad]]$. If $\mathfrak{M}$ and $\mathfrak{M}_1$ are approximately finite (B) and $\chi(\mathfrak{M}) = \chi(\mathfrak{M}_1)$ then $\mathfrak{M}$ and $\mathfrak{M}_1$ are algebraically isomorphic. If $\mathfrak{M}$ is approximately finite (A1) or (A2) and $\chi(\mathfrak{M}) = \aleph_0$ then $\mathfrak{M}$ is approximately finite (B). Extensions of these results hold for AW*-algebras of type II_1 possessing a central trace. (Received January 6, 1956.)

303*t*. Harold Widom: *Embedding in algebras of type* I.

Starting with order convergence in a complete lattice, a notion of convergence is introduced in a commutative AW*-algebra. Using this as a replacement for the ordinary topology on the complex numbers, various topologies are obtained for AW*-modules and AW*-algebras of type I analogous to the usual topologies on Hilbert space and the algebra of bounded operators on Hilbert space. Extending the methods of Dixmier and Feldman to the more general situation, the following results are obtained: Let A be an AW*-algebra with center Z, and assume there exists a complete set of positive linear mappings f: $A \to Z$ satisfying (a) $x \in A$, $a \in Z$ imply $f(ax) = af(x)$, (b) if e_λ are orthogonal projections in A with sup e and $x \in A$ then $f(x^*ex) = \sum f(x^*e_\lambda x)$. Then A is AW*-embeddable in an algebra of type I with center Z. Let B be an algebra of type I with center Z, A an AW*-subalgebra of B containing Z. Then $A = A''$ in B under any of the following conditions: (a) A is of type I, (b) l is the sup of countably many finite projections of A, (c) Z is locally countably decomposable. (Received January 6, 1956.)

304*t*. J. L. Zemmer: *A new class of nonassociative algebras.* Preliminary report.

The class $\mathfrak{G}(F)$ of linear algebras over a field F is defined as follows: $K \in \mathfrak{G}(F)$ if and only if K possesses an invariant bilinear form $h(x, y)$, and $(xy)z - x(yz) = h(y, z)x - h(x, y)z$, for all x, y, z in K. A mapping of $\mathfrak{G}(F)$ onto the set of all associative algebras with identity is defined. This mapping and the equivalence relation determined by it are the main tools used in studying $\mathfrak{G}(F)$. Some of the results are: (1) every right and left ideal in $K \in \mathfrak{G}(F)$ is associative; (2) let $K \in \mathfrak{G}(F)$ have finite dimension over F, if $a \in K$, r, s positive integers, then there exist α_i in F such that $a^r a^s = \alpha_1 a + \alpha_2 a^2 + \cdots + a^{r+s}$, where a^n is defined by $a^1 = a$, $a^n = a^{n-1}a$, for $n > 1$; (3) if $K \in \mathfrak{G}(F)$ is not associative and has finite dimension over F then its radical (as defined by A. A. Albert, *The radical of a non-associative algebra*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 891–897) is its maximal ideal. This last result facilitates the construction of an algebra in $\mathfrak{G}(F)$ whose radical is any preassigned associative algebra with identity. (Received January 11, 1956.)

ANALYSIS

305. Joseph Andrushkiw: *Polynomials with prescribed values at critical points.*

Let D_n, $D_n(x)$, D'_{n-1} be the discriminants of the polynomials $f(z)=t_0+t_1z+\cdots+t_{n-1}z^{n-1}+z^n$, $f(z;x)=f(z)-x$ and their derivative $f'(z)$, respectively. The discriminants are expressed in the form of determinants of the order $2n$, $2n$, $2(n-1)$, respectively, with the coefficients t_k as their elements. D_k, $D_k(x)$, D'_k are their principal diagonal minors of order $2k$. Further, let $a_1, a_2, \cdots, a_{n-1}$ denote the zeros of $f'(z)$. It follows that: 1. $f(a_k)=m_k$, $m_k\neq m_j$, $k\neq j$, $k, j=1, 2, \cdots, n-1$, if and only if $D_n(m_k)=0$; 2. p among m_k equal m if and only if $D_r(m)=0$, $r=n, n-1, \cdots, n-p+1$; 3. m_k represent maxima and minima if 1. (and eventually 2.) and $D'_r>0$, $r=1, 2, \cdots, n-1$; 4. if all maxima equal M and all minima equal m, M and m are the roots of quadratic equation $D_{m+1}(x)=0$ in case $n=2m-1$, and m is the root of linear equation $D_{m+1}(x)=0$ and M a root (namely the immediately larger than m) of quadratic equation $D_{m+2}(x)/(x-m)=0$ in case $n=2m$. (Received December 30, 1955.)

306. R. B. Barrar: *On the inverse Sturm-Liouville problem.*

Consider the differential equation (1) $y''+[\lambda-P(x)]y=0$ for complex $P(x)\in L(0, \pi)$ with the boundary condition (2) $y(0)\cos\alpha+y'(0)\sin\alpha=y(\pi)\cos\beta+y'(\pi)\sin\beta=0$ and (3) $y(0)\cos\alpha+y'(0)\sin\alpha=y(\pi)\cos\gamma+y'(\pi)\sin\gamma=0$; $\sin(\beta-\gamma)\neq 0$. It is shown that if the spectrum of (1) is known for each of the given boundary conditions (2) and (3) then $P(x)$ is uniquely determined. This result is an extension of that given by Levinson for real $P(x)$ in Mat. Tidsskr. B, 1949, pp. 25–30 (1949). The proof utilizes the integral introduced by Gelfand and Levitan for this type of problem [see Levinson, Physical Review vol. 89 (1953) pp. 755–757]. (Received January 12, 1956.)

307. Anatole Beck (p) and J. T. Schwartz: *A vector-valued random ergodic theorem.*

Let $\mathfrak{X}$ be a reflexive B-space and (S, Σ, m) a σ-finite measure space. Let $L_p(S, \Sigma, m, \mathfrak{X})$ denote the space of all strongly measurable functions f defined on S with values in $\mathfrak{X}$ such that $\int_S\|f(s)\|^p m ds<\infty$. Let there be defined on S a strongly measurable function T_s with values in the B-space $B(\mathfrak{X})$ of operators on $\mathfrak{X}$. Suppose that $\|T_s\|\leqq 1$ for all $s\in S$. Let $s\rightarrow h(s)$ be a measure-preserving transformation in (S, Σ, m). Then, for each $X(s)\in L_1(S, \Sigma, m, \mathfrak{X})$, there is a strongly measurable function $\overline{X}(s)\in L_1(S, \Sigma, m, \mathfrak{X})$ such that $\lim_{n\to\infty}(1/n)\sum_{i=1}^{n} T_sT_{h(s)}\cdots T_{h^{i-1}(s)}(X(h^i(s)))=\overline{X}(s)$ a.e. in S, and the limit on the left also exists in the mean of order 1. Moreover, if $m(S)<\infty$, $\overline{X}(s)=T_s\overline{X}(h(s))$ a.e. in S. Under special conditions, it is possible to evaluate the limit $\overline{X}$ and thus derive theorems which are extensions of the Law of Large Numbers and the Mean Ergodic Theorems. (Received January 5, 1956.)

308. H. J. Bremermann: *Holomorphic functionals and complex convexity in Banach spaces.*

A complex-valued functional is "Gateaux holomorphic" in a domain D of a complex Banach space B_c if it is single-valued and its restriction to an arbitrary analytic plane $\{z \mid z=z_0+\lambda a, (z_0\in D, a\in B_c, \lambda$ a complex parameter) is a holomorphic function of λ in the intersection of the plane with D. A notion of holomorphic continuation is defined and a lemma on the simultaneous continuation of holomorphic functionals is proved. A consequence of the lemma is that not every domain is a domain of existence of a holomorphic function (domain of holomorphy). Let $d_D^{(N)}(z)$ denote the distance of the point z from the boundary of D measured in the norm N. Then, if D is a domain of holomorphy, the functional $-\log d_D^{(N)}(z)$ is plurisubharmonic in D. The property of $-\log d_D^{(N)}(z)$ to be plurisubharmonic is invariant with respect to all norms N that generate equivalent topologies. The domains for which $-\log d_D^{(N)}(z)$ is plurisubharmonic are called "pseudo-convex." Some properties of the pseudo-convex domains are studied and compared with those of the convex domains. In particular, tube domains are pseudo-convex if and only if they are convex. (Received January 13, 1956.)

309*t*. R. C. Buck: *A theorem of Bochner and Martin.*

In their book on functions of several complex variables, Bochner and Martin obtain a number of results dealing with formal power series and "inner transformations." This note approaches these from a different point of view. Let P be the linear space of complex polynomials in n variables. The completion of P in a suitable topology τ is $\mathfrak{F}$, the space of formal power series. Let $\mathfrak{L}(\mathfrak{F})$ be the algebra of continuous linear transformations of $\mathfrak{F}$ into itself. Theorem 1. The "inner transformations" are precisely the $T\in\mathfrak{L}$ which are multiplicative. Theorem 2. An inner transformation T has an inverse if and only if the linear part of T is nonsingular. The proof of the latter uses the following simple (and probably well known) criterion: let E be a linear space and $T=I-U$ a transformation in $\mathfrak{L}(E)$; if $\lim U^n x=0$ for all $x\in E$, then T is one-to-one, and its range is dense; if $x+Ux+U^2x+\cdots$ converges for each x, then T is an automorphism of E. (Received January 13, 1956.)

310. R. C. Buck: *Topologies defined by transformations.* Preliminary report.

Let E be a real linear space with a locally convex topology τ. Denote by $\mathfrak{L}(E)$ the algebra of continuous linear transformations T of $\langle E, \tau\rangle$ into itself, and let $\Gamma\subset\mathfrak{L}$. Let $\beta(\Gamma)$ be the smallest topology on E such that each $T\in\Gamma$ is continuous from $\langle E, \beta\rangle$ to $\langle E, \tau\rangle$. Example: Take $E=C(S)$, the space of bounded continuous functions on a locally compact space S, with τ the (normed) topology σ of, uniform convergence on S. Let $A\subset C(S)$ and take Γ as the transformations $x\rightarrow ax$ for $a\in A$. If $A=C_\infty$, the subspace of functions with compact support, then $\beta(\Gamma)=k=$compact-open; if $A=C_0$, the subspace of functions which are zero at infinity, then $\beta(\Gamma)$ becomes the "strict" topology β. Theorem: β and σ have the same bounded sets; $\beta=k$ on (σ) bounded sets; the dual space of $\langle C(S), \beta\rangle$ is $\langle C_\infty, \sigma\rangle^*$. (Received January 13, 1956.)

311. Lamberto Cesari: *Retraction of surfaces.*

A Fréchet continuous (path) surface S: (T, J) of graph $[S]=T(J)$ in E_n and boundary (path) curve θS: (T, J^*) (J a closed Jordan region of boundary J^*) is said to be nondegenerate if no maximal continuum of constancy g for T in J separates J. It is proved that: (I) Given a simple closed curve C in E_n and $\epsilon>0$ there is a $\delta=\delta(C, \epsilon)>0$ such that every surface S with $\|\theta S, C\|<\delta$ has a nondegenerate retraction S_0 with $\|\theta S_0, C\|<\epsilon$ ($\|\quad\|$ Fréchet distance). Given S let g_t denote any continuum g as above (if any) such that gJ^* is not connected. A surface S with $[\theta S]\subset C$ is said to have property P if for every g_t there is at least one arc $\lambda=w_1w_2=\lambda(g)\subset J^*$ with $\lambda\supset gJ^*$, w_1, $w_2\in g$, such that the closed curve c: (T, λ) is relatively homotopic to zero in C, i.e. $c\cong 0(C)$. (II) Every S with $[\theta S]\subset C$ having property P has a retraction S_0

whose boundary θS_0 is freely homotopic to θS in C, i.e., $\theta S_0 \simeq \theta S(C)$, and $[\theta S_0] \subset [\theta S] \subset C$. Also, S_0 is nondegenerate if θS is not nulhomotopic in C. These and other theorems are proved in the author's book *Surface area* (Annals of Mathematics Studies) and are used there in the discussion of the representation problem of continuous mappings (surfaces) from any finitely connected Jordan region, or any plane open set (admissible sets). (Received January 11, 1956.)

312t. Paul Civin and Bertram Yood: *Maximal subalgebras in commutative Banach algebras.*

Let B be a complex commutative B^*-algebra with identity e and space of maximal ideals $\mathfrak{M}$. It is shown that if A is a maximal proper closed subalgebra containing e, then either (1) A distinguished between pairs of points of $\mathfrak{M}$, or (2) $A = \{x \mid x(M_1) = x(M_2)$, where $M_1 \neq M_2$ and $M_i \in \mathfrak{M}\}$. All sets of the type (2) are maximal proper closed subalgebras. If $\mathfrak{M}$ is countable, then any A must be of type (2). Some generalizations are considered. (Received December 2, 1955.)

313. Chandler Davis: *A device for studying Hausdorff moments.*

Numbers μ_k, with $k=1, 2, \cdots$, form a (normalized) Hausdorff sequence provided they can be obtained as $\mu_k = \int x^k dm(x)$, where integration is over $[0, 1]$ and $m([0, 1]) = 1$. Given n numbers $\mu_1, \cdots, \mu_n$, here is a necessary and sufficient condition for them to be the first n members of some Hausdorff sequence. There must exist an $m \times m$ matrix A (where $n = 2m-2$ or $n = 2m-1$), subject to certain requirements, such that $\mu_k = (A^k)_{11}$ for each given μ_k. The requirement on A can be, for instance, that (for $i = 1, \cdots, m$) $A_{ii} = (1-\theta_{i-1})t_i + (1-t_i)\theta_i$, $A_{i,i+1} = A_{i+1,i} = ((1-t_i)(\theta_i - \theta_i^2)t_{i+1})^{1/2}$, and all other $A_{ij} = 0$; with $0 \leqq \theta_i \leqq 1$, $0 \leqq t_i \leqq 1$, and $t_1 = 0$. If only some of $\mu_1, \cdots, \mu_n$ are given, the condition for extendability still holds. If an infinite sequence is given, only the evident alteration is needed: the indices of A must run $1, 2, \cdots$. These easily proved facts yield some of the basic properties of Hausdorff sequences, as well as inequalities between moments which are convenient in statistics. (Received January 12, 1956.)

314t. Albert Edrei: *Logarithmic densities and gap theorems of Pólya.*

Consider the entire function $F(z) = \sum a_n z^n$ and let $\{\lambda_v\}_{v=1}^{\infty}$ be the sequence of all the subscripts $\lambda (\geqq 1)$ such that $a_\lambda \neq 0$. If $L(t) = \sum_{\lambda_v \leqq t} \lambda_v^{-1}$ and $0 \leqq \xi < 1$, the logarithmic density $\Delta(\xi) = \limsup_{x \to +\infty} \{L(x) - L(x^\xi)\}/(1-\xi) \log x$ is analogous to Pólya's density $D(\xi)$ [Math. Zeit. vol. 29 (1929) p. 557]. In particular $\Delta(1) = \lim_{\xi \to 1-0} \Delta(\xi)$ exists and $\Delta(1) \leqq D(0)$. Many of Pólya's results remain true with $D(1)$ replaced by $\Delta(1)$. For instance, let ρ $(\leqq +\infty)$ be the order of $F(z)$ and $\Delta(1)$ the logarithmic density associated with $\{\lambda_v\}$. Then $F(z)$ cannot be of order less than ρ in an angle of magnitude greater than $2\pi\Delta(1)$. (Received January 30, 1956.)

315t. Jacob Feldman: *The uniformly closed ideals in an AW* algebra.*

Let $\mathfrak{A}$ be an AW* algebra, $\mathfrak{L}$ its lattice of projections. All ideals will here be two-sided and uniformly closed. F. Wright has shown a 1-1 correspondence between the ideals of $\mathfrak{A}$ and the equivalence-closed lattice ideals of $\mathfrak{L}$ ("p-ideals"), the ideal $\mathfrak{I}$ corresponding to the p-ideal $\mathfrak{I} \cap \mathfrak{L}$. The inverse correspondence can be given by $\mathfrak{I} = \{A \mid E^{\perp}_{A^*A}(\lambda) \in \mathfrak{I} \cap \mathfrak{L}$ for all $\lambda > 0\}$, E_{A^*A} being the spectral resolution of A^*A. There is a canonical splitup of $\mathfrak{A}$ into a c^* sum of AW* algebras $\{\mathfrak{A}_i\}$. One can completely classify the ideals of $\mathfrak{A}$ by a more or less natural 1-1 correspondence with certain objects associated with the collection $\{Z_i\}$ of centers of the summands. The situation is clearest if $\mathfrak{A}$ is a II_1. Let $\mathcal{Y}=\{Z|Z\in\text{center of }\mathfrak{A},\ 0\leqq Z\leqq 1\}$. For Z_1, Z_2 in $\mathcal{Y}$, say $Z_1\ll Z_2$ if $Z_1\leqq \alpha Z_2$, α real. This induces an equivalence map π on $\mathcal{Y}$, and a natural order among the equivalence classes, making them into a lattice $\mathfrak{L}$. Then if d is the central dimension function on $\mathcal{L}$, $\pi\circ d$ induces a 1-1 order-preserving correspondence between the p-ideals of $\mathcal{L}$ and the lattice-ideals of $\mathfrak{L}$. There are unexpectedly many ideals; for example, for each maximal ideal $\mathfrak{M}$ of $\mathfrak{A}$, the set of ideals under $\mathfrak{M}$ but under no other maximal ideal is nondenumerable. (Received November 3, 1955.)

316. Leonard Gillman: *A continuous exact set.*

A linearly ordered set is *exact* if it is not similar to any of its proper subsets. All examples hitherto have been sets with gaps (Cuesta, Revista Mat. Hisp.-Amer. vol. 14 (1954) pp. 237–268; Ginsburg, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 341–361; Bagemihl-Gillman, Fund. Math. vol. 42.1 (1955) pp. 141–165; and references cited therein). Cuesta (loc. cit.) has asked whether there can exist a *continuous* exact set. In the present paper, the author constructs a continuous exact set A of power $\mathfrak{c}$; the proof of exactness, however, requires the hypothesis that $\mathfrak{c}$ be a *regular* cardinal. The set A is a certain subset of the lexicographically ordered set of all ω-sequences (x_n) of reals; the coordinates x_n are restricted inductively, according to a scheme too complicated to be described here. (Received February 10, 1956.)

317. A. O. Huber: *A characteristic property of polynomials.*

An entire analytic function $w=f(z)$ is a polynomial if and only if there exists a number $\lambda>0$ such that $\int_\sigma |f(z)|^{-\lambda}|dz| = +\infty$ for every locally rectifiable path σ tending to infinity. (Received January 11, 1956.)

318. G. K Kalisch: *On reducing subspaces and unitary invariants of certain Volterra operators.*

If $f\in L=L_p(0,\ 1)$, $T_K f=\int_x^1 K(x,\ y)f(y)dy\in L$ for all $f\in L$, then the subspaces $L_\lambda=L_p(0,\ \lambda)(0\leqq\lambda\leqq 1)$ of L all are reduced by T_K. $\{L_\lambda\}$ is the totality of reduced subspaces if $K\in C^2$, $K(x,\ x)\neq 0$, and if $K(x,\ y)=K(y-x)$ where $K(t)=t^n K_1(t)$, $K_1(0)\neq 0$, $K_1\in C^{n+2}$ in a neighborhood of 0. This theorem is used to find conditions for unitary equivalence of operators T_K of the above kind defined in a Hilbert space. (Received January 13, 1956.)

319. N. D. Kazarinoff: *Asymptotic forms for Whittaker functions with both parameters large.*

The behavior of the Whittaker functions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ for both $|k|$ and $|m|$ large, $|x|$ unrestricted, is studied under the hypothesis that $(m^2-k^2)/k$ be bounded. The variable and parameters may be either real or complex. The analysis consists of three parts. For $|x/2k|>1$, the asymptotic forms are obtained by means of the classical theory of asymptotic solutions of differential equations near an ordinary point, the solutions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ being identified by their behavior as $|x|\to\infty$. Near the turning point $x=2k$ the theory of R. W. McKelvey [Trans. Amer. Math. Soc. vol. 79 (1955) pp. 103–123] is used. For $|x/2k|<1$, use is made of the paper of McKelvey and the author on the asymptotic solutions of differential equations near a regular singular point [Canadian Journal of Mathematics vol. 8 (1956) no. 1]. This research was supported by the USAF through the OSR-ARDC. (Received January 9, 1956.)

320. George Klein: *A theorem on mixed derivatives.*

Let $\Delta_{hk}F(x, y) = F(x+h, y+k) - F(x, y+k) - F(x+h, y) + F(x, y)$. Suppose that everywhere in a closed rectangle R, $\Delta_{hk}F(x, y)/hk$ tends to zero with h^2+k^2. It is shown that this implies that $F(x, y) \equiv \phi(x) + \psi(y)$ in R, and that the argument can be extended to analogous results for functions of several variables. An application of these results establishes the fundamental theorem of calculus for multiple integrals without the use of the concept of uniform continuity. (Received January 13, 1956.)

321t. Erwin Kreyszig: *Decomposition of solutions of partial differential equations.*

Let (1) $w(z, z^*) = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} w_{\mu,\nu}z^{\mu}z^{*\nu}$ be a solution of (2) $w_{zz^*}+a(z,z^*)\, w_z+b(z, z^*)w_{z^*}+c(z, z^*)w=0$, represented by a Bergman operator of the first kind in the form (3)$w(z, z^*) = \exp(-\int_0^{z}a(z,\xi)d\xi)\,[g(z)+\sum_{n=1}^{\infty}q_n(z,z^*)\int_0^{z}\int_0^{z_1} \cdots \int_0^{z_{n-1}}g(\xi)d\xi dz_{n-1} \cdots dz_1]$ (see Bergman, Trans. Amer. Math. Soc. vol. 57 (1945) p. 299 ff.). In (2), a, b, and c are entire functions of two independent complex variables z, z^*. In (3), $q_n(z, 0)=0$, $n=1, 2, \cdots$, and the "associated function" $g(z)$ is analytic and is regular at $z=0$. A particular solution of (2) is called an elementary solution if its associated function is of the form $g(z)=k(z-z_0)^s$, $s=0, \pm 1, \pm 2, \cdots$. Theorem I: $w(z, z^*)$ can be represented by a finite sum of elementary solutions and has at most finitely many singularity planes if and only if the determinants $w_{\sigma}^{(q)} \equiv \det (w_{i,k}^{(q,\sigma)})$, $w_{i,k}^{(q,\sigma)} \equiv w_{i+k+\sigma-2,0}$, $i, k=1, 2, \cdots, q+1$, $\sigma=0, 1, \cdots$ vanish, with the exception of at most finitely many of them. Theorem II: $w(z, z^*)$ possesses exactly one singularity plane (which is a pole surface of first order and, for all values of $z^* \neq 0$, a logarithmic branch surface) if and only if $\limsup_{\mu\to\infty} |w_{\mu,0}w_{\mu+2,0}-w_{\mu+1,0}^2|^{1/\mu} < r^{-2}$ where r denotes the radius of convergence of the power series development of $w(z, 0)$ at $z=0$. (Received February 13, 1956.)

322t. Erwin Kreyszig: *Solutions of partial differential equations satisfying ordinary differential equations.*

As has been recently shown [Journal of Rational Mechanics and Analysis vol. 4 (1955) pp. 907–923] there exist interesting classes of partial differential equations of the form (1) $w_{zz^*}+a(z, z^*)w_z+b(z, z^*)w_{z^*}+c(z, z^*)w=0$ possessing particular solutions which satisfy also an ordinary differential equation in z. From this fact various conclusions on properties of those solutions can be drawn, e.g. concerning their behavior in the neighborhood of singularities etc. It is of interest that the problem of obtaining solutions of (1) satisfying ordinary differential equations can be formulated as a coefficient problem. Theorem: A particular solution $w(z, z^*) = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} w_{\mu,\nu}z^{\mu}z^{*\nu}$ of (1) satisfies, for every fixed value $z^*=z_0^*$, in addition to (1) also an ordinary linear differential equation in z if the determinants $W^{(\sigma)} \equiv \det (W_{ik}^{(\sigma)})$, $W_{ik}^{(\sigma)} = (i+k-2)! \cdot \sum_{\nu=0}^{\infty} w_{i+k-2,\nu}z_0^{*\nu}$, $i, k=1, 2, \cdots, \sigma$; $\sigma=1, 2, \cdots$, are zero, with the exception of at most finitely many of them; the coefficients of that ordinary differential equation are independent of z, but depend on z_0^*, in general. This theorem can be proved by using Bergman operators of the first kind [see Duke Math. J. vol. 6 (1940) p. 537], while in the above-mentioned paper Bergman operators of the third kind [Rec. Mat. vol. 2 (1937) p. 1193] have been used when representing the particular solutions under consideration. (Received February 13, 1956.)

323. R. E. Langer: *On the asymptotic solutions of a class of ordinary differential equations of the fourth order, with special reference to an equation of hydrodynamics.*

The paper is concerned with the differential equation $w^{iv}+\lambda^2[P(z, \lambda)w''+Q(z, \lambda)w'+R(z, \lambda)\omega]=0$, in which P, Q, and R are power series in $1/\lambda$. The matter at issue is the forms of the solutions when $|\lambda|$ is large and the z-region in question contains a simple zero of $P(z, \infty)$ (a turning point). The limiting equation as $\lambda\rightarrow\infty$ in general has a regular singularity at the turning point. When the exponents do not differ by an integer the differential equation is referred to as a regular type. A general theory of the solution forms to arbitary powers of $1/\lambda$ is given. When the exponents differ by an integer the equation is called irregular. There are several categories of irregular equations, and for these, with one exception, the theory is given. The Orr-Sommerfeld equation of hydrodynamics is of this type and is irregular. The application of the general theory to this equation is discussed. (Received January 10, 1956.)

324. Solomon Leader: *Functions integrable with respect to a finitely additive measure.*

Let μ be a finitely additive measure on a Boolean algebra $\mathfrak{A}$ of sets with X as unit. A real-valued function $f(x)$ is integrable if the approximating sums over finite partitions of X converge uniformly with respect to refinement of partition. An integrable function must therefore be bounded except on a set of measure zero. Through consideration of inner and outer measure $\mathfrak{A}$ can be completed in X with respect to μ to obtain the algebra of measurable sets. A function with values 0 and 1 is integrable if, and only if, it is the indicator of a measurable set. From a characterization of the completed algebra it follows that a bounded function is integrable if, and only if, it is a uniform limit of linear combinations of indicators of measurable sets. This result is used to prove Lebesgue's theorem on Riemann integrable functions. (Received December 29, 1955.)

325. J. J. Levin: *On singular perturbations of nonlinear systems of differential equations related to conditional stability.*

Previously announced results [Bull. Amer. Math. Soc. Abstract 61-6-723] are extended to the systems: (1) $dx/dt=f(t, x, y, \epsilon)$, $\epsilon dy/dt=g(t, x, y, \epsilon)$; (2) $dx/dt=f(t, x, y, 0)$, $0=g(t, x, y, 0)$, where x and y are real vectors of m and n components respectively. Where $x=p(t)$, $y=q(t)$ is a solution of (2) for $0\leqq t\leqq T$, it is assumed that k of the characteristic roots of the matrix $g_y(t)=(\partial g_i(t, p(t), q(t), 0)/\partial y_j)$ have negative real parts and that the remaining $n-k$ have positive real parts for $0\leqq t\leqq T$. The existence of a k-dimensional initial manifold in y space, which depends on ϵ and the initial x vector, such that if the initial y vector lies on the manifold then the solution of (1) exists over $0\leqq t\leqq T$ for all sufficiently small $\epsilon>0$ is established. Several properties of this manifold are obtained. The proofs are similar to those of [Abstract 723] and employ some recent results of L. Flatto and N. Levinson. (Received January 10, 1956.)

326t. W. S. Loud: *On a nonlinear differential equation of second order.* Preliminary report.

The differential equation (1) $x''+cx'+g(x)=e(t)$ is considered, where $c>0$, $e(t)$ is continuous with $|e(t)|\leqq E$, $g(x)$ is differentiable with $g'(x)\geqq b>0$ and $g'(x)$ bounded on any finite interval, and $g(0)=0$. It is shown for the more general equation (2) $x''+f(x)x'+g(x)=e(t)$ with $f(x)$ continuous, and $f(x)\geqq c$ that ultimately any solution will satisfy $|x|\leqq Eb^{-1}+4Ec^{-2}$, $|x|\leqq Eb^{-1}+4Ec^{-1}b^{-1/2}$, $|x'|\leqq 4Ec^{-1}$. A growth theorem for solutions of the linear equation (3) $y''+a(t)y=0$ is obtained to the effect that if $|a(t)|\leqq A^2$, then no solution of (3) and no first derivative of a solution can grow more rapidly than exp (At). If $a(t)$ is periodic, this shows that in the Floquet representation of the solution of (3), $y=\exp(\rho t)p_1(t)+\exp(-\rho t)p_2(t)$, $\rho^2\leqq A^2$. The growth theorem is applied to obtain a convergence theorem for solutions of (1). Two solutions are said to converge in case both of $|x_1-x_2|$ and $|x_1'-x_2'|$ approach zero as t becomes infinite. If all solutions of (1) satisfy ultimately $|x|\leqq A$, then all solutions converge in case $c^2>2H(A)$, where $H(A)$ is the maximum of $g'(x)$ for $|x|\leqq A$. If $e(t)$ is periodic, then in this case there is a unique periodic solution of (1) to which all solutions converge. The work for this paper was supported in part by the Office of Ordnance Research. (Received January 11, 1956.)

327. W. S. Loud: *On periodic solutions of second-order linear differential equations with periodic coefficients.*

The differential equation (1) $y''+a(t)y=0$, with $a(t)$ periodic is considered in the case that there exists a periodic solution $p(t)$. It is known in the case of Mathieu's equation that when such a $p(t)$ exists, no other linearly independent solution can be periodic unless $a(t)$ is a constant (Ince, Proc. Cambridge Philos. Soc. vol. 21 (1922) p. 117). Here (1) is studied as a variation equation for (2) $x''+g(x)=e(t)$ with $e(t)$ small at a periodic solution $x_0(t)$ of (3) $x''+g(x)=0$. The variation equation is (4) $y''+g'(x_0(t))y=0$, and $p(t)=x_0'(t)$. If $g(x)$ is nondecreasing in a neighborhood of zero, is differentiable, and satisfies $g(0)=0$, $xg(x)>0$ otherwise, then all solutions of (4) are periodic if and only if $\int_{-B}^{A}[E-G(x)]^{-1/2}[2^{-1}-G(x)g'(x)(g(x))^{-2}]dx=0$, where $G(x)=\int_0^x g(u)du$, E is the constant value of $2^{-1}x_0'^2+G(x_0)$, and A and $-B$ are the maximum and minimum values of x_0. The period of a solution of (3) depends on amplitude in general, and the above condition is equivalent to the period being stationary as a function of amplitude. The case in which $g(x)$ is allowed to vanish in an interval about zero is considered. It is shown also that if $p(t)$ has no zeros, (1) can not have all solutions periodic, and that if $p(t)$ is a trigonometric polynomial, no other solution of (1) is, unless $a(t)$ is a constant. The work for this paper was supported in part by the Office of Ordnance Research. (Received January 11, 1956.)

328. M. A. Martino: *Applications of a certain measure of non-analyticity.*

Let D be a simply-connected domain in the complex plane, and L the set of Lebesgue-summable complex-valued functions on D. If $f\in L$, and $I=[a_1, b_1; a_2, b_2]$ is a rectangle in D, then $J_f(I)=\oint_{(I)}f(z)dz$ is well-defined provided none of the coordinates, a_1, b_1, a_2, b_2, belong to a certain set N of measure zero. J_f can be linearly extended to a functional of finite unions of nonoverlapping rectangles, all of whose side-coordinates do not belong to N. If the exceptional set N can be chosen so the resulting functional is bounded, then J_f extends to a complex-valued measure μ_f. Let B be the subspace of those functions in L for which μ_f is so obtainable. Letting A be the subspace of those functions equal to an analytic function on D, almost everywhere, the factor space B/A is isomorphic with the linear space of complex-valued bounded measures on D. For each $f\in B$, $f(z)=a(z)-(1/2\pi i)\int\int_D d\mu_f/(\zeta-z)$, for some $a\in A$. This representation allows function-theoretic inferences for f from assumptions on μ_f. Examples: An extension of Vitali's compactness theorem follows from Egoroff's theorem. For continuous $f(z)$, the existence of $\int\int_D d\mu_f/|\zeta-z_0|^2$ implies $f(z)$ is approximately derivable at z_0. (Received January 11, 1956.)

329. C. J. Neugebauer: *A characterization of the Lebesgue area.*

Let A be an admissible set of the Euclidean plane E_2, i.e., A is either an open subset of E_2 or a finite union R of disjoint finitely connected Jordan regions or an open subset of R. Denote by $\mathfrak{T}$ the class of all continuous mappings (T, A) from an admissible set $A\subset E_2$ into the Euclidean 3-space E_3. A mapping $(T, R)\in\mathfrak{T}$ will be termed elementary provided it is Fréchet equivalent to a quasi-linear mapping. Let now $\mathfrak{F}$ be the class of all functionals $\Phi(T, A)$ defined for each $(T, A)\in\mathfrak{T}$ such that (a) $\Phi(T, A)$ is non-negative; (b) $\Phi(T, A)$ is lower-semicontinuous; (c) $\Phi(T, A)$ satisfies the Kolmogoroff principle; (d) $\Phi(T, A')\leqq\Phi(T, A)$, if A' is an admissible subset of A; (e) $\Phi(T, A)$ agrees with the Lebesgue area $L(T, A)$ for elementary mappings. For each $\Phi\in\mathfrak{F}$, denote by $\mathfrak{T}(\Phi)$ the class of all mappings $(T, A)\in\mathfrak{T}$ for which $\Phi(T, A) = L(T, A)$, and let $\mathfrak{T}^* = \bigcap\mathfrak{T}(\Phi)$, $\Phi\in\mathfrak{F}$. Let now $\mathfrak{T}'$ be the class of mappings $(T, A)\in\mathfrak{T}$ satisfying: for every $\epsilon>0$ there exists an elementary mapping (T', R), $R\subset A$, such that (1) $|T'(w)-T'(w')|\leqq|T(w)-T(w')|$, $w, w'\in R$, (2) $|L(T, A)-L(T', R)|<\epsilon$, if $L(T, A)<\infty$; $L(T', R)>\epsilon^{-1}$, if $L(T, A)=+\infty$. It is proved that $\mathfrak{T}^*=\mathfrak{T}'$. If the condition (e) is replaced by (e') $\Phi(T, A)$ agrees with the Lebesgue area for all quasi-linear mappings, then an example of a functional Φ is given which satisfies (a), (b), (c), (d), and (e') but which does not coincide with the Lebesgue area. (Received January 12, 1956.)

330*t*. Johannes Nitsche: *On solutions of differential equations of mixed type.*

The analytic solutions of $U_{xx}+x^pK(x)U_{yy}=0$ are considered in the region where x^p is real ($p>0$) and $K(x)=1+\sum_1^\infty k_nx^n>0$. Putting $\xi=\int_0^x[x^pK(x)]^{1/2}dx$, $\eta=y$, $N(\xi)=(1+2\gamma_1)[1+\sum_1^\infty N_n\xi^{2n\gamma_2}]/4\xi$ and $\zeta=\xi+i\eta$ one obtains $U_{\zeta\bar\zeta}+N(\xi)(U_\zeta+U_{\bar\zeta})=0$ ($\gamma_j=(-1)^j/(p+2)$). The solutions U can be represented as real parts of functions $u(\zeta, \bar\zeta)\equiv\int_{-1}^{+1}E(\zeta, \bar\zeta, \tau)\cdot f(\zeta(1-\tau^2)/2)(1-\tau^2)^{-1/2}d\tau$ (cf. S. Bergman, Amer. J. Math. vol. 70, p. 856; vol. 74, p. 444), $f(\sigma)$ an arbitrary analytic function of one complex variable, integration along $|\tau|=1$. The integraloperator E has the form $E(\zeta, \bar\zeta, \tau) =\exp[-2\int^\xi N(\xi)d\xi]\sum_{j=1}^2[\zeta(1-\tau^2)/2]^{\gamma_j+1/2}G^{(j)}(s, t)$. Here $s=2\xi/\zeta\tau^2$, $t=\zeta\tau^2$ and $G^{(j)}$ are solutions of a system of integral equations with $G^{(j)}(s, 0)=s^{\gamma_j+1/2}F(\gamma_j+1/2, \gamma_j+1, 2\gamma_j+1; s)$ (F hypergeometric series). They converge for all t and $|s|<1$, that is, in the sector S: $\{3\xi^2<\eta^2\}$ containing the parabolic line, and can be analytically continued along each path avoiding $s>0$. The properties of $f(\sigma)$ permit statements about the properties of the corresponding solution, independent of the special form of $x^pK(x)$. If e.g. $f(\sigma)$ has a pole of mth order at σ_0, then $u(\zeta, \bar\zeta)$ is regular in the intersection of S and $|\zeta|<|2\sigma_0|$, and near $\zeta=2\sigma_0$ has a development $\sum_{\nu=1}^{2m-1}P_\nu(\zeta, \bar\zeta)\cdot(\zeta-2\sigma_0)^{-\nu/2}$+reg. function. So (analogous to S. Bergman, ZaMM vol. 32, pp. 33) many theorems of the theory of analytic functions can be transferred to solutions $u(\zeta, \bar\zeta)$ resp. $U(x, y)$. (Received January 16, 1956.)

331*t*. Johannes Nitsche: *On solutions of Cauchy's problem for equations of mixed type with data on the parabolic line.*

The author considers the analytic solutions of $U_{xx}+x^PK(x)U_{yy}=0$ satisfying initial conditions $U(0, y)=\sum_0^\infty \alpha_ny^n$, $U_x(0, y)=\sum_0^\infty\beta_ny^n$ (in the region where x^P is real ($p>0$) and $K(x)=1+\sum_1^\infty k_nx^n>0$). Using the notation and the results of the preceding abstract these solutions can be represented as the real part of $\int_{-1}^{+1} E(\zeta, \bar{\zeta}, \tau) \cdot f(\bar{\zeta}(1-\tau^2)/2)(1-\tau^2)^{-1/2} d\tau$. For the coefficients a_n of $f(\sigma)=\sum_0^\infty a_n\sigma^n$ one obtains an expression, which asymptotically has the form $(i/2)^n a_n = A\alpha_n n^{\gamma_2}[1+O(n^{2\gamma_1})] + B\beta_n n^{\gamma_1}[1+O(n^{-1})]$ where A and B are complex constants depending only on p. So from the behavior of the power series $\sum \alpha_n y^n$ and $\sum \beta_n y^n$ one can infer the properties of the solution, especially the region of its regularity and the type of its singularities. The analogy of the Hadamard-Mandelbrojt classification of the functions of one complex variable by means of their singularities can also be developed. (Concerning the case of elliptic equations compare S. Bergman, Trans. Amer. Math. Soc. vol. 57, p. 299.) If, for instance, the α_n, β_n have the property that $|(a_{n+1}/a_n)-(1/a)| < C\cdot\rho^n$, $\rho<1$ and the point a is situated in the sector S then the solution is regular in the intersection of S and $|\zeta|<|2a|$ and on $|\zeta|=|2a|$ has only a single singularity which is of the type Re $[P(\zeta, \bar{\zeta})(\zeta-2a)^{-1/2}]$+reg. function. (Received January 16, 1956.)

332. L. A. Rubel: *An optimal gap theorem.*

Theorem. Given a sequence $\{p_n\}$ of positive integers, in order that there exist a non-null function $f(z)$ which has a power series expansion of the form $f(z)=\sum a_n z^{p_n}$ and whose region of regularity includes the full negative real axis $-\infty \leq z \leq 0$, it is necessary and sufficient that lim inf $n/p_n>0$. The theorem is proved by translating the problem into one on entire functions by using an interpolation theorem of Buck [Duke Math. J. vol. 13 (1946) pp. 541–549]. The corresponding problem on entire functions has recently been solved by the author. [Bull. Amer. Math. Soc. Abstract 61-3-417]. (Received January 10, 1956.)

333*t*. Morris Schreiber: *Representation of non-normal operators.*

Pursuing the investigation reported in [Bull. Amer. Math. Soc. Abstract 62-2-290] and following the terminology of that abstract, it is shown that for operators A with $\|A\| \leq 1$ whose operator measure F is equivalent to Lebesgue measure on the unit circle C and whose spectrum contains C, the map $f \to f(A)=\int_C f(t)dF(t)$ is isometric and isomorphic between the algebra of continuous boundary values of functions analytic for $|z|<1$, with supremum norm, and the uniform closure of the algebra generated by A and I. An example of such an operator is the unilateral shift operator on sequence space. (Received January 16, 1956.)

334*t*. Morris Schreiber: *Unitary dilations.*

If A is a contraction ($\|A\| \leq 1$) on a Hilbert space H and U is a unitary operator on a larger space $K \supset H$ such that (i) $PU^n x = A^n x$ for all n and $x \in H$, where P is the projection onto H, and (ii) no subspace of $K \ominus H$ reduces U, then it follows that U is uniquely determined within unitary equivalence by A, and it is known that every contraction does determine such a unitary operator [Nagy, Acta Szeged vol. 15 (1953) pp. 87–92. See also the author's abstract, Bull. Amer. Math. Soc. Abstract 62-2-290]. It shall be called "The" unitary dilation $U(A)$ of A. The structure of $U(A)$ is determined in the following cases. (1) If $\|A\|<1$ and dim $(H)=n \leq \aleph_0$, then $U(A)$ is the n-fold copy of the bilateral shift operator on sequence space. (2) If A is a projection with dim $(A)=m$, codim $(A)=n$, then $U(A)$ is the direct sum of the m-dimensional identity operator and the n-fold copy of the bilateral shift operator on sequence space. From these results it follows that (3) all A with $\|A\|<1$ on separable space have unitarily equivalent unitary dilations, and (4) two projections P and Q are unitarily equivalent if and only if $U(P)$ and $U(Q)$ are. (Received January 16, 1956.)

335*t*. Maurice Sion: *Variational measure.*

Let F be a family of sets, σF their union, and $\mathcal{P}(F)$ the set of all finite, disjointed subfamilies P with $\sigma P = \sigma F$. For f a function on σF and ν a measure on rng f, the variational measure $\mu = V(F, f, \nu)$ is defined by: $\mu(A) = \sup_{P \in \mathcal{P}(F)} \sum_{\alpha \in P} \nu(f(A\alpha))$, for $A \subset \sigma F$. If σF is a topological space and F satisfies certain conditions, $f(\alpha)$ is ν-measurable for $\alpha \in F$ and $f^{-1}\{y\}$ is closed for ν-almost all y, A is μ-measurable, and for every $A' \subset A$ with $\mu(A') > 0$ there is $B \subset A'$ with $0 < \mu(B) < \infty$, then $f(A)$ is ν-measurable. As an application, if $\mathfrak{M}_0$ denotes the set of outer measures on the line under which intervals are measurable, f is real-valued, continuous on the irrationals and $f^{-1}\{y\}$ is countable for all y, then f maps a set, ν-measurable for all $\nu \in \mathfrak{M}_0$, into one of the same kind. Let $\mathfrak{M}_1$ be the set of all $\nu \in \mathfrak{M}_0$ for which there is no sequence S such that: S_n is a finite, disjointed family; $\sigma S_n = \sigma S_0$; S_{n+1} is a refinement of S_n; $\nu(\sigma S_0) > 0$; if $B \subset \sigma S_0$ and $\nu(B) > 0$ then $\sum_{A \in S_n} \nu(AB) \to^n \infty$. Then a real-valued function continuous on the irrationals maps a set, ν-measurable, for all $\nu \in \mathfrak{M}_1$, into one of the same kind. (Received January 9, 1956.)

336*t*. Mary C. Weiss: *On the Littlewood series* $\sum n^{-1/2} \exp(i\beta n \log n + in\theta)$.

The series (denote it by S and its partial sums by S_N) exhibits a number of phenomena resembling those of series of independent random variables. The following two properties are typical: (1) for almost every θ we have $\limsup |S_N| / \{\log n \cdot \log\log\log\log n\}^{1/2} = \alpha$ where $\alpha = \alpha(\beta)$ is a finite positive constant, (2) On every set $E \subset (0, 2\pi)$ and of positive measure, the real and imaginary parts of S_N obey asymptotically (for $N \to \infty$) independent Gaussian Laws with mean value 0 and dispersion $C(\log n)^{1/2}$. Similar results apply to Abel means and to a family of series related to S. (January 26, 1956.)

337*t*. Mary C. Weiss: *The law of the iterated logarithm for lacunary trigonometric series.*

Let $S_k(x)$ denote the partial sums of a lacunary series (*) $\sum_1^\infty \rho_k \cos(n_k x + \alpha_k)$, $n_{k+1}/n_k > q > 1$, and let $B_k^2 = 1/2 \sum_1^N \rho_j^2$. Salem and Zygmund have shown that (**) $\limsup |S_N(x)| / \{2B_N^2 \log \log B_N\}^{1/2} \leqq 1$ with the provision that (***) $\text{Max}_{1 \leqq k \leqq N} \rho_k = o\{B_N/(\log \log B_N)^{1/2}\}$. Recently Erdös and Gal proved the inequality opposite to (*) for the series $\sum_1^\infty \exp i(n_k x + \alpha_k)$. It can be shown that the inequality opposite to (**) for the series (*) satisfies the general condition (***). (Received January 26, 1956.)

338. Albert Wilansky: *The inverse matrix in summability.*

Let $'A$, A', A^{-1} denote any matrices such that $'AA = AA' = AA^{-1} = (A^{-1})A = I$; A being a fixed conservative $(=K)$ matrix. THEOREM. *If* $\|A^{-1}\| < \infty$ (i.e. A is K_r) *then* A^{-1} *is conservative.* Here $\|A\| = \sup_n \sum_k |a_{nk}|$. Known Tauberian theorems follow easily, e.g. $a_{nk} \leqq 0$ for $k < n$, $\neq 0$ for $k = n$, $= 0$ for $k > n$, imply A equivalent to convergence: for these hypotheses imply A^{-1} has non-negative entries. Therefore $\|A^{-1}\| < \infty$. Examples show the theorem false for $'A$ and A'; e.g. A may be multiplicative 0 yet $\|A'\| < \infty$ (but A co-null implies A' not conservative and $\|'A\| = \infty$.) Other results: If A is row-finite, sums no divergent sequences, then A^{-1} is conservative if it exists; there exists regular triangle A summing divergent sequences with regular $'A$; and a regular triangle A summing no divergent sequence with $\|'A\| = \infty$. If A is reversible and $\|A'\| < \infty$ then A sums no divergent sequence, A^{-1} exists and is conservative. If A is reversible, not necessarily conservative, then A^{-1} exists iff $c_n \equiv 0$ (see Banach, p. 50). (Received January 6, 1956.)

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339. R. J. Arms, L. D. Gates, Jr. (p), and Bernd Zondek: *Successive block overrelaxation.*

This generalizes D. Young's result on the "successive overrelaxation" method for solving linear systems (Trans. Amer. Math. Soc. vol. 76 (1954) p. 72). Let a linear system with matrix A be given by (1) $\sum_{j=i}^{N} A_{ij}\bar{x}_j=\bar{y}_i$ $(i=1, 2, \cdots, N)$. A_{ij} are blocks of A, with $A_{ii}\neq 0$, $\bar{x}_i$ and $\bar{y}_i$ vectors whose orders equal that of A_{ii}. Definition: Matrix A has *block property A* with respect to the partition $A=(A_{ij})$ if there exist disjoint sets S and T whose sum is the set of first N integers, such that if $A_{ij}\neq 0$ and $i\neq j$, then $i\in S$, $j\in T$ or $j\in S$, $i\in T$. An iterative algorithm analogous to Young's designated as *successive block over-relaxation* (SBOR) follows from (1) and the results of Young's paper hold in generalized form for the new method, using Block Property A. An application of the SBOR method is to elliptic difference systems where the components of x_i are unknown values on the ith grid line, allowing A_{ii} to be easily inverted. It has also been shown that in the usual situation, the SBOR method converges faster than the SOR method. (Received January 11, 1956.)

340t. A. A. Blank: *Axiomatics of the binocular visual geometry.*

Since Luneburg's work, *The mathematical analysis of binocular vision* (Princeton, 1947), it has generally been conceded on insufficient physical evidence that the visual sensory space may be taken as metric and even homogeneous. In the present work there is exhibited a set of axioms sufficient to sustain these conjectures. The axioms are qualitative or nonmetric in nature and have already been verified by the experiments of Hardy, Rand, and Rittler in collaboration with the author. (Received December 29, 1955.)

341t. Margaret F. Conroy: *Stresses at the boundary of a symmetrically shaped hole in an infinite plate.*

In this paper the elastic stresses at the boundary of a symmetrically shaped hole in an infinite plate, due to bands of force or concentrated forces applied normal to the boundary and in the plane of the plate, are determined. The class of hole shapes considered are those contours in the z-plane into which the unit circle in the ζ-plane is mapped by the function $z=\zeta[A_0+\sum_{n=1}^{n=m} A_n\zeta^{-kn}]$ where $k\geqq 2$ and the $A_n s$ are real coefficients restricted so that the roots of $dz/d\zeta$ lie within the unit circle in the ζ-plane. The force distribution has an angular periodicity of $2\pi/k$. The method of solution consists of a combination of the conformal mapping technique and the Mushelisvili method for solving plane elasticity problems. The results obtained are applicable to finite plates of such an extent that the stresses at the outer boundary due to the loading at the inner boundary, or hole contour, vanish. The results presented in this paper were obtained in the course of research conducted at the Watertown Arsenal Laboratory, Watertown, Mass. (Received January 13, 1956.)

342. J. B. Diaz and G. S. S. Ludford (p): *On the integration methods of Bergman and Le Roux.*

In a previous note (Quarterly of Applied Mathematics vol. 12 (1955) pp. 422–427) a correspondence was found between the two representations of solutions $u(x, y)$ of the linear hyperbolic equation $u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0$ given by the integral operator methods of S. Bergman, Math. Sbornik N.S. vol. 2 (1937) pp. 1169–1198, and J. Le Roux, Ann. École Norm. (3) vol. 12 (1915) pp. 227–316. This correspondence has the disadvantage that to some regular Bergman kernels $E(x, y, t)$ there may correspond singular Le Roux kernels $U(x, y, \alpha)$. The purpose of the present note is to give an alternative correspondence which avoids this difficulty. An extension and an example are given. (Received January 11, 1956.)

343. Peter Henrici: *Remark on the quotient-difference algorithm of H. Rutishauser.*

Let $F(\lambda)=\sum_{n=0}^{\infty} a_n\lambda^{-n-1}$ be meromorphic in $|\lambda|>\Lambda$ with simple poles at the points λ_i $(i=1, 2, \cdots)$, $|\lambda_1|>|\lambda_2|>\cdots>\Lambda$. Let $H_k^{(n)}=\det(a_{ij})$, where $a_{ij}=a_{n+i+j-2}$ $(i, j=1, 2, \cdots, k)$, and $H_k^{(n)}(\lambda)=\det(b_{ij})$, where $b_{ij}=a_{n+i+j-2}$ $(i=1, 2, \cdots, k+1;$ $j=1, 2, \cdots, k)$ and $b_{i,k+1}=\lambda^{i-1}$ $(i=1, 2, \cdots, k+1)$. Put $p_k^{(n)}(\lambda)=H_k^{(n)}(\lambda)/H_k^{(n)}$. It is known that for $n\to\infty$ and fixed k, $p_k^{(n)}(\lambda)=\prod_{i=1}^{k}(\lambda-\lambda_i)+O((\lambda'_{k+1}/\lambda_k)^n)$, where $\lambda'_{k+1}=|\lambda_{k+1}|+\epsilon$, $\epsilon>0$ arbitrary. Lemma: For $\lambda=\lambda_i$ $(i<k)$ this relation can be sharpened as follows: $p_k^{(n)}(\lambda_i)=O((\lambda_{k+1}/\lambda_i)^n)$, where λ_{k+1} is the same as before. This yields a simple proof of the following theorem of H. Rutishauser (Z. Angew. Math. Physik vol. 5 (1955) p. 389): Let A be a matrix of order N with simple eigenvalues λ_i, $|\lambda_1|>|\lambda_2|>\cdots>|\lambda_N|>0$. For $n=0, 1, \cdots$ let $x_k^{(n)}$, $y_k^{(n)}$ $(k=0, 1, \cdots, N-1)$ be the vectors obtained by biorthogonalising the two systems $A^{n+k}x$, $(A^T)^{n+k}y$ $(k=0, 1, \cdots, N-1)$, where x and y are two almost arbitrary vectors. Then, as $n\to\infty$, $x_k^{(n)}/|x_k^{(n)}|\to v_{k+1}$, $y_k^{(n)}/|y_k^{(n)}|\to w_{k+1}$, where v_i and w_i are the eigenvectors of A and A^T belonging to λ_i. (Received January 11, 1956.)

344. J. R. Isbell (p) and W. H. Marlow: *Hotspot problems.*

Several authors have considered the problem: determine a number v and optimal strategies such that the game with payoff matrix $(a_{ij}+b_{ij}v)$, each $b_{ij}<0$, has value zero. (See Shapley, Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 1095–1100). The present starting point is such a game, due to C. B. Tompkins (George Washington University Logistics Papers, Issue 2, 1950) concerning assignment of military forces to a disputed area, "Hotspot." It is noted that there is a formal connection (via the description of combat) between the matrix game, $(-a_{ij}/b_{ij})$ and Tompkins' game; furthermore, whenever one has a solution in pure strategies then so does the other, with the same good strategies and value. For the second simplification of Tompkins' game, rather than having new assignments at stated intervals, changes are permitted whenever a unit on either side is destroyed. This new game has a solution in pure strategies; hence neither player would wish to change his assignment at any other time than those prescribed. The best strategy for either player in any case is to assign all, none, or exactly one of his units to Hotspot. Research supported (in part) by the Office of Naval Research. (Received January 12, 1956.)

345*t*. Erwin Kreyszig: *On the Fresnel integrals for complex argument.*

The Fresnel integrals $C(z)=2^{-1}\int_0^z t^{-1/2}\cos t\,dt$, $S(z)=2^{-1}\int_0^z t^{-1/2}\sin t\,dt$ are of importance in connection with many problems in physics and technics. The zeroes $z_n=x_n+iy_n$ of $C(z)$ and $z_n^*=x_n^*+iy_n^*$ of $S(z)$ are asymptotically located on one and the same logarithmic curve, in alternating order; this curve can be represented in the form $y=\pm 2^{-1}\log 2\pi x$. For numerical calculation of these zeroes the following approximation formulas are useful: $x_n\sim(4n-1)\pi/2-\log\pi(4n-1)^{1/2}/(4n-1)\pi$, $y_n\sim\log\pi(4n-1)^{1/2}$, $x_n^*\sim 2n\pi-\log 2\pi n^{1/2}/4n\pi$, $y_n^*\sim\log 2\pi n^{1/2}$, $n=1, 2, \cdots$. The functions $z^{1/2}C(z)$ and $z^{1/2}S(z)$ are entire functions of order 1 of divergence class as follows from the distribution of the zeroes. The Weierstrassian products are

$C(z)=z^{1/2}\prod_{n=1}^{\infty}(1-z^2/z_n^2)(1-z^2/\bar{z}_n^2)$, $S(z)=3^{-1}z^{3/2}\prod_{n=1}^{\infty}(1-z^2/z_n^{*2})(1-z^2/\bar{z}_n^{*2})$. (Received February 13, 1956.)

346. Walter Littman: *On the existence of gravity waves near critical speed.*

A new proof is given for the existence of two-dimensional gravity waves near critical speed, using a method originally devised by Friedrichs and Hyers for the purpose of proving the existence of solitary waves. The problem is first reduced to solving a nonlinear boundary value problem for the potential equation in an infinite horizontal strip, with boundary conditions which are nonlinear on top and linear on the bottom. This problem, in turn, is reduced to that of solving a nonlinear integral equation. The essential step in the procedure is a stretching of the independent variable $\hat{\phi}=a\phi$. There results a one parameter family of integral equations all depending on a. For $a=0$ the equation becomes very simple, and is, in fact, solved explicitly, the solution (which contains two free parameters) giving rise to the first approximation of the sought-after flow, and agreeing essentially with the approximate description of J. B. Keller. By perturbing about this solution, the original integral equation is solved for small positive a. This amounts to proving the existence of periodic flows for sufficiently large wavelengths and for near-critical speeds. This method yields both sub- and supercritical flows. (Received December 30, 1955.)

347. Imanuel Marx: *On the characteristic values of a complex quadratic form.*

For a (*non-Hermitean*) diagonalizable complex quadratic form $Q=(Az, z)$ (*complex symmetric inner product*), each characteristic value is a stationary value of $(Az, z)/(z, z)$, or equivalently of (Az, z) for vectors z such that $(z, z)=1$, and stationary vectors are characteristic vectors (Comm. Pure Appl. Math. vol. 7 (1954) p. 625). For an arbitrary complex quadratic form, each characteristic value whose characteristic vector lies in an m-dimensional maximal cyclic invariant subspace relative to A is a stationary value of $|(Az_1, z_1)+\cdots+(Az_m, z_m)|/|(z_1, z_1)+\cdots+(z_m, z_m)|$ subject to the conditions $(z_1, z_1)=(z_2, z_2)=\cdots=(z_m, z_m)$ and $(z_j, z_k)=0$ $(j\neq k)$, or equivalently of $(Az_1, z_1)+\cdots+(Az_m, z_m)$ for an orthonormal set. Every orthonormal basis of the invariant subspace furnishes a solution of the variational problem. Each isotropic characteristic vector z makes (Az, z) stationary subject to the condition $(z, z)=0$. All characteristic values and corresponding invariant subspaces and characteristic vectors can be determined by the solution of a succession of variational problems in suitably chosen subspaces of admissible vectors. (Received December 27, 1955.)

348. L. E. Payne: *Inequalities for eigenvalues of a free plate.*

Upper bounds for the eigenvalues Λ_i of a free plate may be obtained by application of the Rayleigh-Ritz technique in the minimum principle. However, there appears to be no practical method for estimating Λ_i from below. In this paper the following lower bound for Λ_i is obtained: $\Lambda_{i+2}\geqq(1-\sigma)\mu_2\mu_i$, ($\sigma$ is Poisson's ratio), where μ_i are the eigenvalues of the free membrane of the same shape. (The plate equation has been taken in the form $\Delta^2u-\Lambda u=0$, where Δ denotes the Laplace operator.) If the region possesses certain symmetry properties it is known that μ_2 may be bounded from below, the bound depending only on the geometry of the region (see Payne and Weinberger, Tech. Note BN-56, University of Maryland). One obtains in this case a lower bound for the first nonzero eigenvalue of the free plate which depends only on the geometry of the plate. (Received January 12, 1956.)

349. I. F. Ritter: *A compact method for solving a system of linear equations of any condition.*

A compact method is described for solving the equation $AX=R$ for the vector or matrix X by triangularization of the numerical matrix A. In this method, an upper triangle C is determined so that $AC=B$ is a lower triangle, the *post*-multiplication of A by the explicitly computed C being in contrast to the *pre*-multiplication of A in the Gauss elimination. Decisive advantages over other compact methods based on factoring A are secured by the following features of the method employing C explicitly: (1) The right side R enters into the computations only after B and C are established and are available for a convenient error correction process. (2) With back-substitution avoided by the use of the Aitken "below-the-line" bookkeeping device, the numerically largest component of each nonzero column of B can be chosen as pivot. (3) Columns of C corresponding to zero-columns of B represent the complete solution of the homogeneous system $A\bar{x}=0$. (4) The error in any pair of corresponding columns of B and C can be corrected immediately upon the construction of these columns. (5) With any loss in significant digits avoided by the error correction, the solution of a nearly singular, or ill-conditioned, system can be computed as accurately as desired. (6) If A is modified slightly, the pivot columns need not be adjusted for solving the new system. (7) Features (3), (4), and (6) form the basis of an effective approximation method for solving the characteristic value problem $A(\lambda)\bar{y}(\lambda)=0$ with a matrix $A(\lambda)$ not necessarily linear in the parameter λ. (Received January 9, 1956.)

350*t*. H. E. Salzer: *Numerical integration of $y''=\phi(x, y, y')$ using osculatory interpolation.*

This present paper presents several new procedures, with formulas, for the stepwise integration of $y''=\phi(x, y, y')$, starting from some initial values of y, y' and y''. These methods are based upon different types of osculatory interpolation formulas. One method uses an approximation to the next y'' by a formula for osculatory extrapolation for the derivative, and then finds the next y' and y using formulas for $n\frac{1}{2}$-point osculatory interpolation and $(n+1)$-point osculatory quadrature respectively. Another method is applied to the special case of $y''=\phi(x, y)$, employing some new interpolation formulas based upon the function and its second derivative. Finally, for $y''=\phi(x, y, y')$ the most accurate set of formulas utilizes "hyperosculatory" interpolation, which is based upon the function with its first and second derivatives. Predicting formulas, which are purely extrapolatory, give the next y and y' (then y''). Refining formulas employ the next y'' in "$n\frac{1}{3}$-point" hyperosculatory formulas to find a new y', and then they employ y'' and this new y' in "$n\frac{2}{3}$-point" hyperosculatory formulas to find a new y (then a new y''). This refining process may be repeated if necessary. (Received November 23, 1955.)

Geometry

351. Louis Auslander (p) and Masatake Kuranishi: *On the holonomy group of locally Euclidean spaces.*

Let $M^n(\pi)$ be a compact n dimensional locally euclidean space. (A space is called locally euclidean if it has a Riemann metric with curvature and torsion zero.) Then Bieberbach (Math. Ann. vol. 70 (1911) pp. 297–336) proved that the holonomy group $h(\pi)$ of $M(\pi)$ is a finite group. The authors have recently proven, using the methods of cohomology theory of groups, that any finite group G can be faithfully represented as the holonomy group of a compact n dimensional locally euclidean space for sufficiently large n. (Received December 19, 1955.)

352t. James Eells, Jr.: *The differential geometry of mapping spaces. I.*

For the purposes described in II, a detailed study of differentiable manifolds $\mathfrak{M}$ of (possibly) infinite dimension is made; i.e., each point of $\mathfrak{M}$ has a neighborhood U (called a *coordinate patch*) homeomorphic with an open set in a complete locally convex linear space (real or complex), and coordinate transformations are required to be of class C^k in the sense of Hildebrandt-Graves (Trans. Amer. Math. Soc. vol. 29, pp. 127–153) (with certain extensions). Tangent vectors are defined, and the formalities of tensor bundles and Grassmann bundles are carried over to the infinite dimensional case by means of the topologized tensor product; completeness of the tangent spaces and of their tensor products is used essentially. Using the method of sheaves, we prove a de Rham theorem relating the real (singular) cohomology of $\mathfrak{M}$ to that derived from the exterior algebra of differential forms on $\mathfrak{M}$. If $\mathfrak{M}$ is locally a Hilbert space, the sheaf of germs C^k forms on $\mathfrak{M}$ is fine as usual; otherwise, the proof of de Rham's theorem requires more detailed study of the integration of differential forms. Also, a study is made of the relation of the cohomology of $\mathfrak{M}$ to that of the nerves of certain open covers of $\mathfrak{M}$. (Received January 11, 1956.)

353. James Eells, Jr.: *The differential geometry of mapping spaces. II.*

Let M be a C^k Riemannian manifold with metric ρ, and let S be a compact Hausdorff space. The totality of continuous maps $x:S\rightarrow M$ form a function space $\mathfrak{M}$ metrized by $R(x, y) = \sup \{\rho(x(s), y(s)) : s\in S\}$. Using the Riemannian structure of M (both the metric and directional properties of geodesics are used) it is shown that $\mathfrak{M}$ has a differentiable structure (in the sense of I) of class C^{k-1}. The coordinate relations are made by considering the geodesic equations on M as functions of the initial conditions. If M is a Lie group, then the function space $\mathfrak{M}$ is an analytic group (of infinite dimension, in general); relations to the associated Lie algebra are considered. In the case that $\mathfrak{M}$ is the space of paths on M with fixed origin m_0, then the space of loops at m_0 is a closed submanifold of $\mathfrak{M}$; Serre's fibre space over M is then studied with cohomology elements represented by differential forms. Several variants of the manifold $\mathfrak{M}$ are developed; e.g., spaces of differentiable maps and of absolutely continuous maps, spaces of mappings of a compact pair (S, S_0) into (M, M_0), where M_0 is a closed submanifold of M. (Received January 11, 1956.)

354. S. I. Goldberg: *Projectively Euclidean Hermitian manifolds. I.*

In an Hermitian manifold M^n a curve $C:\xi^i = \xi^i(t)$ is called a geodesic if the tangent vector $\dot{\xi}^i$ at a point remains tangent under a parallel displacement along C. It can be shown that the n.a.s. conditions are $\ddot{\xi}^\alpha + \Gamma_{\beta\gamma}^{\alpha}\dot{\xi}^\beta\dot{\xi}^\gamma + 2g_{\gamma\rho^*}g^{\alpha\sigma^*}S_{\tau^*\sigma^*\rho^*}\dot{\xi}^{\tau^*}\dot{\xi}^\gamma = f(t)\dot{\xi}^\alpha$. (For definitions and notation cf. the author's paper *Tensorfields and curvature in Hermitian manifolds with torsion*, Ann. of Math., to appear). In analogy to the Riemannian case, it is asked if it is possible to transform the connection Γ_{jk}^i of M^n in such a way that the geodesics remain geodesics. A group G_{SP} of "special" projective transformations $\Gamma_{jk}^i \rightarrow \Gamma_{jk}^{\prime i}$ is defined as follows: (i) the $\Gamma_{jr}^{\prime i}$ are self-adjoint and vanish for indices of opposite parity, (ii) $S_{jk}^{\prime i} = S_{jk}^i$. If under a "special" projective transformation M^n is mapped into a locally Euclidean space, M^n is said to be G_{SP}-flat. It is shown that an Hermitian manifold is G_{SP}-flat if and only if it has "constant" holomorphic curvature. This generalizes a result due to S. Bochner (*Curvature in Hermitian metric*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 179–195. Bochner's theorem, as well as its generalization, is a complex analogue of a well-known theorem due to H. Weyl (*Zur Infinitesimalgeometrie: Einordnung der projectiven und der konformen Auffassung*, Göttinger Nachrichten (1921) pp. 99–112). (Received January 9, 1956.)

355*t*. S. I. Goldberg: *Projectively Euclidean Hermitian manifolds*. II.

This is a continuation of the author's previous abstract in this bulletin. The following additional results are obtained: 1. The only manifolds whose geodesics correspond to the geodesics of a manifold of "constant" holomorphic curvature are manifolds of "constant" holomorphic curvature. This is the complex analogue of a well-known theorem due to Beltrami. 2. A G_{SP}-flat manifold is conformal with some Kaehler (Einstein) space. For projectively flat manifolds this result is trivial. 3. A G_{SP}-flat compact manifold is projectively flat. This last statement is proved with the help of a well-known lemma due to S. Bochner. (Received January 9, 1956.)

356. C. C. Hsiung: *On the differential geometry of hypersurfaces in the large*.

Let V^n (V^{*n}) be an orientable hypersurface of class C^3 imbedded in a Euclidean space E^{n+1} of dimension $n+1\geqq 3$ with a closed boundary V^{n-1} (V^{*n-1}) of dimension $n-1$. Suppose that there is a one-to-one correspondence between the points of V^n, V^{*n} such that at corresponding points V^n, V^{*n} have the same normal vectors. Let M_α ($\alpha=1, \cdots, n$) be the αth mean curvature of V^n at a point P defined by $\binom{n}{\alpha}M_\alpha = \sum\kappa_1\cdots\kappa_\alpha$, where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of V^n at P, and the expression on the right side is the αth elementary symmetric function of $\kappa_1, \cdots, \kappa_n$. Let dA be the area element of V^n at P, and p^* the oriented distance from a fixed point O in E^{n+1} to the tangent hyperplane of V^{*n} at the point P^*, which corresponds to P under the given correspondence. The purpose of this paper is first to derive some expressions for the integrals $\int_{V^n} M_\alpha p^* dA$ ($\alpha=1, \cdots, n$), and then using one of these expressions to prove that under some further conditions V^n and V^{*n} are congruent or symmetric. (Received January 11, 1956.)

357*t*. Detlef Laugwitz: *Riemannian metrics associated with convex bodies and normed spaces*.

It is well known that with a bounded convex body C containing the origin O of a real vector space L as an inner point there is associated a convex positive definite functional $F(x)$ (the norm or Minkowski metric). Recently, E. R. Lorch has begun the study of another geometry closely related to C, namely the Riemannian geometry defined by $g_{ik}(x)=2^{-1}\partial^2F^2(x)/\partial x^i\partial x^k$ (See Bull. Amer. Math. Soc. Abstract 622-247). The results are valid for spaces of finite as well as of infinite dimension, for the latter cases by the use of the authors notational convention (See Math. Zeit. vol. 61 (1954) pp. 100–118). The curvature tensor R_{ijkl} of this geometry as computed by Lorch and the tensor S_{ijkl} of the tangent Minkowski space in Cartan's theory of Finsler spaces are related by $R=F^2S$. *For two-dimensional spaces, R vanishes identically; this implies $S=0$ for any two-dimensional Finsler space.* A necessary and sufficient condition that all the straight lines of L be geodesics for g_{ik} is that C be an ellipsoid with 0 as its center, in other words, that the Minkowski metric F be euclidean. (Received January 11, 1956.)

358*t*. T. S. Motzkin: *L'Hospital's rule for nonzero characteristic.* Preliminary report.

An algebroid branch at the origin of affine n-space over a ground field of characteristic p, given by a reduced representation $x=\sum_{k=1}^{\infty} a_k t^k$, $a_k=(a_{1k}, \cdots, a_{nk})$ (i.e. not similarly representable by any parameter $\tau=\sum_{k=2}^{\infty} c_k t^k$), has a straight line of highest intersection multiplicity or "0-tangent" $x=a_{k_0}t$, where k_0 is the smallest k with $a_k \neq 0$; a "1-tangent" or specialized generic tangent $x=a_{k_1}t$, $k_1=\min k$, $ka_k\neq 0$; a "λ-tangent" $x=a_{k_\lambda}t$, $k_\lambda=\min k$, $C_{k_2}, a_k \neq 0$. These definitions are independent of the reduced representation selected. There may be up to k_0-q+2 different tangents, where $q=\prod(r_j+1)$ and the r_j are the digits of k_0 to base p. (Received January 13, 1956.)

359. D. K. Pease: *Four-space representation of the complex plane.* I. *The line at infinity.*

If (X, Y) is a point of the complex plane, C_2, with $X=x+iu$, $Y=y+iv$, let (x, y, u, v) be the corresponding point of R_4. A complex line of C_2 is represented by a *regular plane* of R_4. The regular planes of R_4 intersect R_3^∞, the hyperplane at infinity of R_4, in *regular lines*, which comprise the real lines of a linear congruence determined by the *regular directrices* D_1: $x+iu=y+iv=0$ and D_2: $x-iu=y-iv=0$. To each point of C_1^∞, the complex line at infinity of C_2, there corresponds a regular line of R_3^∞. Any selected real plane, R_2^∞, of R_3^∞ contains a unique point of each regular line except for the unique regular line lying in R_2^∞. Since this regular line is equivalent to one point of C_1^∞, R_2^∞ is topologically equivalent to a complete Argand plane. This confirms a conjecture of J. S. Taylor. (Received January 6, 1956.)

360*t*. D. K. Pease: *Four-space representation of the complex plane.* II. *Affinities.*

An affinity, $X=AX'+BY'+E$, $Y=CX'+DY'+F$, on C_2 induces a *type* A affinity on R_4. (See part I for notation and terms.) The type A affinities are just those real affinities on R_4 for which the regular directrices, D_1 and D_2, are invariant. If the characteristic roots of the collineation induced on C_1^∞ are P and Q, those on R_3^∞ are P, Q, $\bar{P}$, $\bar{Q}$. The following is a list of canonical forms together with the Segre characteristics. $\{\quad\}$ indicates complex conjugates. $\widehat{\ }$ indexes the root $\rho=1$ corresponding to the fixed R_3^∞. P and Q are complex; p and q are real. Non-parabolic: $X=PX'$, $Y=QY'$, $[\{11\}\{11\}\hat{1}]$; $X+PX'$, $Y=qY'$, $[(11)\{11\}\hat{1}]$; $X=PX'$, $Y=PY'$, $[\{(11)(11)\}\hat{1}]$; $X=PX'$, $Y=\bar{P}Y'$, $[\{(11)(11)\}\hat{1}]$; $X=PX'$, $Y=Y'$, $[(11\hat{1})\{11\}]$; $X=PX'$, $Y=Y'+F$, $[(\hat{2}1)\{11\}]$; $X=pX'$, $Y=qY'$, $[(11)(11)\hat{1}]$; $X=pX'$, $Y=pY'$, $[(1111)\hat{1}]$; $X=pX'$, $Y=Y'$, $[(11\hat{1})(11)]$; $X=pX'$, $Y=Y'+F$, $[(\hat{2}1)(11)]$; $X=X'$, $Y=Y'$, $[(1111\hat{1})]$; $X=X'$, $Y=Y'+F$, $[(\hat{2}111)]$. Parabolic: $X=PX'+Y'$, $Y=PY'$, $[\{22\}\hat{1}]$; $X=pX'+Y'$, $Y=pY'$, $[(22)\hat{1}]$; $X=X'+Y'$, $Y=Y'$, $[(22\hat{1})]$; $X=X'+Y'$, $Y=Y'+F$, $[(\hat{3}2)]$. Of the 69 types of real affinities on R_4, 15 occur as type A. (Received January 6, 1956.)

361*t*. D. K. Pease: *Four-space representation of the complex plane.* III. *Anti-affinities.*

An anti-affinity, $X=A\bar{X}'+B\bar{Y}'+E$, $Y=C\bar{X}'+D\bar{Y}'+F$, on C_2 induces a *type* B affinity on R_4. (See parts I and II for notation and terms.) The type B affinities are just those real affinities on R_4 for which the regular directrices, D_1 and D_2, are interchanged. The type A and type B affinities together comprise the *regular affinities* on R_4. The following is a list of the canonical forms together with the Segre characteristics. Hyperbolic: $X=p\bar{X}'$, $Y=q\bar{Y}'$, $[1111\hat{1}]$; $X=p\bar{X}'$, $Y=\bar{Y}'$, $[(1\hat{1})111]$; $X=p\bar{X}'$, $Y=p\bar{Y}'$, $[(11)(11)\hat{1}]$; $X=\bar{X}'$, $Y=\bar{Y}'$, $[(11\hat{1})(11)]$; $X=p\bar{X}'$, $Y=\bar{Y}'+F$, $[\hat{2}111]$; $X=\bar{X}'$, $Y=\bar{Y}'+F$, $[(\widehat{21})(11)]$. Parabolic: $X=p\bar{X}'+\bar{Y}'$, $Y=p\bar{Y}'$, $[22\hat{1}]$; $X=\bar{X}'+\bar{Y}'$, $Y=\bar{Y}'$, $[(2\hat{1})2]$; $X=\bar{X}'+\bar{Y}'$, $Y=\bar{Y}'+F$, $[\widehat{32}]$. Elliptic: $X=Q\bar{Y}'$, $Y=P\bar{X}'$, $[\{11\}\{11\}\hat{1}]$; $P\bar{Q}$ real, $[\{(11)(11)\}\hat{1}]$; $P\bar{Q}=1$, $[(11\hat{1})(11)]$; $X=Q\bar{Y}'+E$, $Y=P\bar{X}'$, $P\bar{Q}=1$, $[(\widehat{21})(11)]$. Of the 69 types of real affinities on R_4, eight occur as type B and 17 as regular affinities. (Received January 6, 1956.)

Logic and Foundations

362*t*. R. M. Friedberg: *The solution of Post's problem by the construction of two recursively enumerable sets of incomparable degrees of unsolvability.*

Kleene and Post have constructed (*The upper semi-lattice of degrees of recursive unsolvability*, Ann. of Math. vol. 59 (1954) pp. 379–407, esp. pp. 385–390) two functions β_1 and β_2 of incomparable degrees. In the present paper this construction is made recursive so that β_1 and β_2 are characteristic functions of recursively enumerable sets. Functions α_1 and α_2 and conditions $\{2\}$ and $\{3\}$ are dropped. (Notation here and below is from Kleene-Post, *op. cit.*) Each condition $\{0, e\}$ or $\{1, e\}$ cannot be treated in one step since the choice between subcases cannot be made effectively. Therefore treatment of condition $\{0, e_1\}$ may require indefinite indecision as to the value of $\beta_1(a)$, while treatment of $\{1, e_2\}$ requires immediate decision. In the present construction, if $e_1 \leqq e_2$, condition $\{1, e_2\}$ is treated on the assumption that $\beta_1(a)$ will never be set equal to 0; this treatment is afterwards corrected if condition $\{0, e_1\}$ requires $\beta_1(a)=0$. If $e_2 < e_1$, condition $\{1, e_2\}$ is treated fully, and treatment of $\{0, e_1\}$ is shifted from a to another argument. Similarly for β_2. It is shown that the treatment of each condition is shifted or corrected only a finite number of times. (Received January 10, 1956.)

363*t*. Richard Montague: *Zermelo-Fraenkel set theory is not a finite extension of Zermelo set theory.*

By Zermelo-Fraenkel set theory (ZF) is understood the theory, formulated in the lower predicate calculus with identity, whose only non-logical symbol is the two-place predicate '$\in$' and whose non-logical axioms are (1) the axiom of extensionality, (2) the power-set axiom, (3) the sum-set axiom, (4) the axiom of infinity, (5) the axiom of replacement (which comprises every instance of the schema '$(x, y, z)((x\in a\cdot\phi(x, y)\cdot\phi(x, z))\supset y=z)\supset(\exists b)(y)(y\in b\equiv(\exists x)(x\in a\cdot\phi(x, y)))$'), and (6) the Fundierungsaxiom. For precise formulations of (1)–(4), (6), and the pair-axiom see Mostowski, Fund. Math. vol. 37 (1950), p. 111. Zermelo set theory (Z) differs from ZF by containing, in place of (5), the pair-axiom and the Aussonderungsaxiom (the schema '$(\exists x)(y)(y\in x\equiv(\phi\cdot y\in a))$'). McNaughton (Proc. Amer. Math. Soc. vol. 5 (1954) pp. 505–509) has shown that neither theory is finitely axiomatizable. Further, it is well known that ZF is an extension of Z. It can be shown, however, that ZF, if consistent, is not equivalent to any theory obtained by adding finitely many axioms to Z; that is, ZF is not a *finite extension* of Z. In fact, a stronger result holds: if T is any consistent extension without new constants of ZF, then T is not a finite extension of Z. (Received November 28, 1955.)

364t. Richard Montague and Donald Kalish: *A simplification of Tarski's formulation of the predicate calculus.*

Tarski (Bull. Amer. Math. Soc. Abstract 57-1-74) has given the following formulation, which does not employ the notion of free variable or that of proper substitution on free variables, of the lower predicate calculus with identity. Axiom-schemata: (1) If ϕ is tautologous, then ϕ is an axiom. (2) $\ulcorner(\alpha)(\beta)\phi\supset(\beta)(\alpha)\phi\urcorner$ is an axiom. (3) $\ulcorner(\alpha)(\phi\supset\psi)\supset((\alpha)\phi\supset(\alpha)\psi)\urcorner$ is an axiom. (4) $\ulcorner\phi\supset(\alpha)\phi\urcorner$ is an axiom if α does not occur in ϕ. (5) $\ulcorner(\alpha)\phi\supset\phi\urcorner$ is an axiom. (6) $\ulcorner\sim(\alpha)\sim\alpha=\beta\urcorner$ is an axiom. (7) $\ulcorner\alpha=\beta\supset(\phi\supset\psi)\urcorner$ is an axiom if ϕ is atomic and ψ is obtained from ϕ by replacing an occurrence of α by β. Rules: Modus Ponens, Generalization. Schemata (2) and (5) can be eliminated; the resulting system is complete and independent. The Rule of Generalization can be eliminated by taking as axioms all universal generalizations of the axioms comprised under (1), (3), (4), (6), (7); the resulting system is also complete and independent. (Received November 28, 1955.)

365t. Richard Montague and Donald Kalish: *An extension of Tarski's notion of satisfaction.*

Let L be a language, i.e., a set of predicates and operation symbols. Let $\langle DxRFA\rangle$ be a model of L; i.e., let $x\in D$, R and F be functions such that if π is an n-place predicate in L then $R(\pi)$ is a set of n-tuples of members of D, and if δ is an n-place operation symbol in L then $F(\delta)$ is a function from the class of n-tuples of members of D into D, and A be a function from the class of variables into D. Then, by simultaneous recursion: (i) if α is a variable, α is a *term* (of L) and *val* (α) (for $\langle DxRFA\rangle$) is $A(\alpha)$; (ii) if δ is an n-place operation symbol in L and $\zeta_1, \cdots, \zeta_n$ are *terms*, then $\ulcorner\delta\zeta_1, \cdots, \zeta_n\urcorner$ is a *term* and *val* $(\ulcorner\delta\zeta_1 \cdots \zeta_n\urcorner)$ is $[F(\delta)]$ (*val* $(\zeta_1) \cdots$ *val* (ζ_n)); (iii) if ϕ is a *formula* (of L), then $\ulcorner\imath\alpha\phi\urcorner$ is a *term* and *val* $(\ulcorner\imath\alpha\phi\urcorner)$ is the unique z such that $\langle DxRFA^{\alpha}_{z}\rangle$ *satisfies* ϕ if there is such a z, and x otherwise (where A^{α}_{z} is that function whose value for α is z and which otherwise coincides with A); (iv) if π is an n-place predicate in L, ζ, η, $\zeta_1, \cdots, \zeta_n$ are *terms*, and ϕ, ψ are *formulas*, then $\ulcorner\zeta=\eta\urcorner$, $\ulcorner\pi\zeta_1 \cdots \zeta_n\urcorner$, $\ulcorner(\alpha)\phi\urcorner$, $\ulcorner\sim\phi\urcorner$, $\ulcorner(\phi\supset\psi)\urcorner$, etc. are *formulas*, and $\langle DxRFA\rangle$ *satisfies* each of them under the usual conditions. (Received December 13, 1955.)

366t. Richard Montague and Donald Kalish: *Formulations of the predicate calculus with operation symbols and descriptive phrases.*

For terminology see abstract 365. In abstract 364 is given a simplified formulation of the lower predicate calculus with identity for languages which do not contain operation symbols. A completely analogous formulation can be given for an arbitrary language L. Axiom schemata (here ϕ, ψ, and ζ, η are respectively formulas and terms of L without descriptive phrases): (1), (3), (4) of abstract 1; (6′) $\ulcorner\sim(\alpha)\sim\alpha=\zeta\urcorner$, where α does not occur in ζ; (7′) $\ulcorner\zeta=\eta\supset(\phi\supset\psi)\urcorner$, where ϕ is a quasi-atomic formula of L (i.e., $\ulcorner\zeta_1=\zeta_2\urcorner$ or $\ulcorner\pi\zeta_1 \cdots \zeta_n\urcorner$), and ψ is obtained from ϕ by replacing an occurrence of ζ by η. Rules: Modus Ponens, Generalization. This system is complete: if ϕ is a formula of L without descriptive phrases, then ϕ is a theorem if and only if ϕ is satisfied by every model of L. To extend this formulation to include descriptive phrases, omit the restriction which excludes them from formulas and terms, and add (for β not occurring in $\ulcorner\imath\alpha\phi\urcorner$): (8) $\ulcorner(\alpha)(\phi\equiv\alpha=\beta)\supset\beta=\imath\alpha\phi\urcorner$; (9) $\ulcorner(\beta)\sim(\alpha)\cdot(\phi\equiv\alpha=\beta)\supset\imath\alpha\phi=\imath\gamma\gamma=\gamma\urcorner$. This system also is complete and is formulated without the notions of free variables and proper substitution. (Received December 13, 1955.)

367*t*. J. P. Roth: *An algorithm for the problem of Quine.*

It has been seen that a truth-functional formula ϕ defines a cubical complex K (*A combinatorial topological method for the synthesis of switching systems in n variables*, Bull. Amer. Math. Soc. Abstract 62-2-281), any equivalent formula yielding an isomorphic complex; the problem of Quine (Amer. Math. Monthly vol. 62 (1955) pp. 627–631), finding a shortest normal equivalent, is interpreted as that of finding a *cover* consisting of q_k k-cubes of K of various dimensions k such that the sum $\sum_1^n q_k(n-k)$ is a minimum. An algorithm designed for automatic machine computation yields covers which e.g. cannot be improved by Quine's subsuming and consensus operations. Elaborations over previous results for going below the Quine minimum are also given. (Received January 10, 1956.)

368*t*. A. R. Schweitzer: *Mathematics and literary composition.* I.

This paper is concerned with a composition "C" consisting of a set of statements "A," the foundation of C, and a set of statements "B" elaborating A in accordance with a plan of thought "D" leading to a concept or system or concepts. Two types of the composition C are considered, namely, mathematical composition C_1 (Mathematics) and literary composition C_2 (literature) understood as a play or a novel or a poem. Corresponding to C_1, A is a set of postulates, B is the set of theorems derived from A in accordance with D, a system of logic. (Received January 9, 1956.)

369*t*. A. R. Schweitzer: *Mathematics and literary composition.* II.

In continuation of the preceding abstract; corresponding to C_2 A is a plot, B elaborates A in accordance with D, a system of literary technique. Examples are given of each type of composition; in particular for C_2 the author selects Shakespearean plays including King Lear, Hamlet, Othello, Macbeth, Merchant of Venice leading respectively to the concepts ingratitude, indecision, jealousy, ambition, inhumanity. Reference is made to Martin Johnson, *Art and scientific thought*, New York, 1949; G. D. Birkhoff, *Aesthetic measure*, Collected Mathematical Papers, vol. III, p. 382; Paul Goodman, *The structure of literature*, Chicago, 1954. The author interprets Goodman's treatise as a study of the whole-part relation with application to the literature in the sense indicated in the preceding abstract. (Received January 9, 1956.)

370*t*. A. R. Schweitzer: *Mathematics and literary composition.* III.

An example of the analogy of mathematics with literary composition is found in the occurrence of the betweenness relation in the foundations of geometry and in the use, in effect, of this relation by Aristotle in his treatise on poetry (Poetics: beginning, middle, ending). (Received January 9, 1956.)

371*t*. R. L. Vaught: *On the axiom of choice and some metamathematical theorems.*

Consider the following three known theorems concerning the existence of models of sentences (of the first-order predicate logic): (1) a sentence having a model of power $\mathfrak{b}$ has also a model of every power $\mathfrak{a}$ such that $\aleph_0 \leqq \mathfrak{a} \leqq \mathfrak{b}$; (2) a sentence having a model of power $\aleph_0$ has also a model of every power $\mathfrak{c} \geqq \aleph_0$; (3) if Q is a set of sentences, in which the set of individual constants involved may have an arbitrary power $\mathfrak{d}$, and every finite subset of Q has a model, then Q has a model, whose power is not greater than $\mathfrak{d}+\aleph_0$. Theorem: *Each of* (1), (2), *and* (3) *implies the Axiom of Choice in its general* *form*. As regards (3), this answers a question, raised by Henkin (Trans. Amer. Math. Soc. vol. 74 (1953) p. 420), as explicitly formulated. (On the other hand, the statement, obtained from (3) by dropping the final phrase ("whose . . . ") has been shown by Henkin and Łos to be equivalent to the existence of prime ideals in Boolean algebras, and its exact relationship to the Axiom of Choice is still not determined.) The Theorem is derived from the known result (of Tarski) that the Axiom of Choice is implied by the statement: if $\aleph_0 \leqq \mathfrak{b}$, then $\mathfrak{b}^2 = \mathfrak{b}$. (Received January 13, 1956.)

Statistics and Probability

372. D. G. Austin: *On second derivatives of Markoff functions—denumerable case.* Preliminary report.

The author has shown that if i and j are stable states in a Markoff process with conditional transition functions $p_{ij}(t)$, $i, j = 1, 2, \cdots$, then each of the functions $p_{kj}(t)$ and $p_{ik}(t)$ $(k = 1, 2, \cdots)$ possess continuous first derivatives (Tech. Report AF 18(600)-760). The inequality $D_*^{(2)} p_{ij}(t+s) \geqq \sum_k Dp_{ik}(t)Dp_{kj}(s)$ is verified. It is shown that $Dp_{ij}(t)$ is absolutely continuous and that, for almost all points (t, s), (*) $D^2 p_{ij}(t+s) = \sum_k Dp_{ik}(t)Dp_{kj}(s)$. If for some s the $Dp_{kj}(s)$ are bounded then $Dp_{ij}(t)$ possesses a continuous derivative on the interval $t \geqq s$. In the case where all states are stable the following result is obtained: (the notation of Doob is used): If $\sum_{j \neq i} q_{ij} = q_i$ and $\sum_j q_{ij} q_j$ converges, then the $p_{ij}(t)$ $(j = 1, 2, \cdots)$ possess finite continuous second order derivatives on the interval $0 \leqq t < \infty$ which satisfy the differential equation (*); if in addition $\sum_j q_{ij} q_j^2$ converge the $p_{ij}(s)$ possess continuous third order derivatives on the infinite interval. (Received January 13, 1956.)

373t. A. A. Blank: *The existence and uniqueness of a uniformly most powerful unbiased randomized test for the binomial.*

The Neyman-Pearson theory of tests was utilized by K. D. Tocher in Biometrika vol. 37 (1950) for discrete distributions to obtain necessary conditions for the uniformly most powerful tests among unbiased randomized tests. In particular, he exhibited the necessary form of such tests for the binomial. A proof of existence and uniqueness for a test of the requisite form is exhibited herein. It is a corollary that the test satisfies the stated criteria. (Received December 29, 1955.)

374. R. V. S. Chacon: *Markov transition probability functions which are not necessarily measurable.*

Let $p_{ij}(t)$ denote the transition probability functions of a denumerable Markov process, with no further assumptions. It is proved that $\liminf_{t\to\infty} p_{ij}(t) \leqq \liminf_{t\to\infty} p_{jj}(t)$ and $\liminf_{t\to\infty} p_{ij}(t) = \liminf_{t\to\infty} p_{jj}(t)$ or $p_{ji}(t) \equiv 0$ and also the same equations for the limit superior, with no reversal of the inequalities. Using this theorem it is proved that a necessary and sufficient condition that $\lim_{t\to\infty} p_{ij}(t)$ exist for all i, j is that $\lim_{t\to\infty} p_{kk}(t)$ exist for all k. It is also proved that if the $p_{ij}(t)$ are so that each is either positive or identically zero, then $\lim_{t\to\infty} p_{ij}(t)$ exists for all i, j. (Received February 1, 1956.)

375t. D. G. Kendall and G. E. H. Reuter: *The determination of the ergodic projection for Markov chains and processes.* I.

Let $(p_{ij}: i, j = 0, 1, 2, \cdots)$ be the one-step transition probabilities for a Markov chain with a countable infinity of states; let N (N_+) be the set of vectors (positive vectors) x such that $\sum_\alpha |x_\alpha| < \infty$ and $x_j = \sum_\alpha x_\alpha p_{\alpha j}$ (all j); and let $N^*(N_+^*)$ be the set of vectors (positive vectors) y such that $\sup_\alpha |y_\alpha| < \infty$ and $y_i = \sum_\alpha p_{i\alpha} y_\alpha$ (all i). A state j is "positive" if and only if it lies in the support of an element of N; two positive states are in different "classes" if and only if there exists an element of N whose support contains one and not the other. Let the positive state j be in the class C^ρ; then among the elements of N_+ such that $x_j = 1$ there is a least, x^j, and $\pi^\rho \equiv x^j/\|x^j\|$ depends only on C^ρ and has C^ρ as its exact support. Also among the elements of N_+^* such that $(y, \pi^\rho) = 1$ there is a least, $\breve{\omega}^\rho$, say. Theorem: $\pi_{ij} \equiv (C, 1) \lim_{n\to\infty} p_{ij}^n = (\breve{\omega}^\rho)_i(\pi^\rho)_j$ for all i. (Of course $\pi_{ij} = 0$ for all i when j is not a positive state.) (Received December 19, 1955.)

376*t*. D. G. Kendall and G. E. H. Reuter: *The determination of the ergodic projection for Markov chains and processes.* II.

(1) If $(P_t : t \geqq 0)$ is the transition semigroup of operators on the Banach space l associated with a Markov process having transition probabilities $p_{ij}(t)$ then the above construction gives $\pi_{ij} \equiv \lim_{t\to\infty} p_{ij}(t)$ when N (N_+) is the set of (positive) vectors annihilated by the infinitesimal generator Ω and N^* (N_+^*) is the set of (positive) vectors annihilated by the adjoint operator Ω^*. (2) If $Q \equiv (q_{ij}: i, j = 0, 1, 2, \cdots)$ is a matrix of finite real numbers nonnegative save on the principal diagonal and having zero row-sums and if the Markov process defined by $p'_{ij}(0+) = q_{ij}$ is unique, then the problem of determining the π's when the q's are known is solved as follows. Write N_+ for the set of positive vectors x such that $\sum_\alpha x_\alpha < \infty$ and $\sum_\alpha x_\alpha q_{\alpha j} = 0$ (all j), and write N_+^* for the set of positive vectors y such that $\sup_\alpha y_\alpha < \infty$ and $\sum_\alpha q_{i\alpha} y_\alpha = 0$ (all i). The positive states and classes can be found as before but now the supports of the elements of N_+ are to be used. The minimality properties determining π^ρ and $\breve{\omega}^\rho$ as elements of the new N_+ and N_+^* still hold and we have as a solution to the problem the Theorem: $\pi_{ij} \equiv \lim_{t\to\infty} p_{ij}(t) = (\breve{\omega}^\rho)_i(\pi^\rho)_j$ for all i when $j \in C^\rho$, and $\pi_{ij} = 0$ for all i when j is not a positive state. (Received December 19, 1955.)

377. Emanuel Parzen: *A law of large numbers for identically distributed random variables with identically distributed increments.*

Let the random variables $X(t)$, defined for t in $T = \{0, \pm 1, \pm 2, \cdots\}$, be identically distributed, with common distribution function $F(x)$ and characteristic function $\phi(u)$. For every τ, let the increments $X(t+\tau) - X(t)$ be identically distributed, with characteristic function $\phi(u; \tau)$. Then for every Borel function g, defined on the real line, such that $E|g| = \int_R |g(x)| dF(x) < \infty$, the sample means $M_n(g)$, defined by $(n+1)M_n(g) = \sum_{t=0}^n g(X(t))$, converges as a limit in first mean, and if $E|g|^2 < \infty$, then the $M_n(g)$ converge as a limit in quadratic mean. If, further, for every real u, $(n+1)^{-1}\sum_{t=0}^n \phi(u; t) \to |\phi(u)|^2$ as $n \sim \infty$, then $M_n(g) \to Eg = \int_R g(x) dF(x)$ as a limit in first mean if $E|g| < \infty$, and as a limit in quadratic mean if $E|g|^2 < \infty$. (Received January 13, 1956.)

Topology

378. R. D. Anderson: *A continuous curve admitting monotone open maps onto all locally connected metric continua.*

The universal curve M is a one-dimensional continuous curve which is an analog of the Cantor middle third set and is obtainable by punching rectangular "holes" out of a cube in a particular regular fashion. In a forthcoming paper the author characterizes the universal curve. The principal theorem of this paper asserts that for any locally connected continuum S there exists a monotone open map f of M onto S with the additional property that for each $s \in S$, $f^{-1}(s)$ is homeomorphic to M. It has not previously been known whether any continuum admits either monotone or open maps onto all locally connected continua. (Received January 11, 1956.)

379t. Steve Armentrout: *On spirals in the plane.*

It is shown that there exist in the plane an arc M, a point O not belonging to M, and a collection G of arcs such that (1) every arc of G has O as one end point and some point of M as its other end point and no two arcs of G have any point, other than O, in common, (2) if X is a point of M, there is an arc of G which contains X and spirals down on X, (3) if g is an arc of G and P is a point such that g spirals down on P, then P belongs to M, (4) G^* is bounded. (Received December 19, 1955.)

380t. Hubert Arnold: *Closed level curves.*

Denote by $\bar{D}$ the closed disk $x^2+y^2 \leqq a^2$ and by L and D its circumference and interior. Denote by $f(p)$ the transformation (single- or multiple-valued) $u=u(x, y)$, $v=v(x, y)$ from $\bar{D}$ of the $x-y$ plane into the $u-v$ plane. Assume $f(p)$ continuous for p in $\bar{D}$ and locally a homeomorphism for p in D, and assume that $f(p)=(0, 0)$ if and only if $p=(0, 0)$. Then the set of points (x, y) for which $|f(p)|^2=u^2+v^2=k^2$ is a single simple closed continuous curve, for each constant k for which $0<k<\min_{p\in L}|f(p)|$. (Received January 11, 1956.)

381t. Bernhard Banaschewski: *On the Weierstrass-Stone approximation theorem.*

By the Weierstrass-Stone approximation theorem, any ring R of bounded continuous real valued functions on a compact space K containing (i) the constants and (ii) sufficiently many functions (that is: to any x, $y \neq x$ in E an f such that $fx \neq fy$) is dense in the ring of all bounded continuous real valued functions on K relative to uniform convergence. An example is given, obtained by refining the topology of the Čech compactification βE of an arbitrary completely regular E at the points of $\beta E - E$ without change of the topology of the subspace E, which shows that the validity of this theorem extends beyond the category of compact spaces. In fact, the following is proved: Let E have sufficiently many bounded continuous real valued functions. Then, the Weierstrass-Stone approximation theorem holds for E if and only if any completely regular filter $\mathfrak{A}$ on E has nonvoid intersection $\bigcap A$, $A \in \mathfrak{A}$. (Complete regularity of $\mathfrak{A}$: $\mathfrak{A}$ has a basis $\mathfrak{B}$ of open sets such that for any $B \in \mathfrak{B}$ there exists a $B' \subseteq B$ in $\mathfrak{B}$ and a continuous function f mapping E onto the real unit interval, vanishing on B' and equal to 1 outside B.) In particular: If the approximation theorem holds for a completely regular E, then E is compact. (Received January 12, 1956.)

382. R. H. Crowell: *Invariants of alternating link types.*

If $\Delta(t_1, \cdots, t_\mu)$ is the Alexander polynomial of a tame link type L of multiplicity μ, the *reduced Alexander polynomial* $\tilde{\Delta}(t)$ of L is equal to: $\Delta(t)$ if $\mu=1$; $(t-1)\Delta(t, \cdots, t)$ if $\mu \geqq 2$. It is known that degree $\tilde{\Delta}(t) \leqq 2h+\mu-1$, where h is the genus of L (G. Torres, *On the Alexander Polynomial*, Ann. of Math. vol. 57 (1953)). A link type L is *admissible* iff there exists a connected, alternating, directed link projection P of type L. Three theorems are proved: (i) For any admissible link type L of multiplicity μ and reduced Alexander polynomial $\tilde{\Delta}(t)$, degree $\tilde{\Delta}(t)=2h+\mu-1$. For admissible link types then, the genus is an effectively calculable invariant. Included in the proof of (i) is a proof of: (ii) The coefficients of the reduced Alexander polynomial of an alternating link type alternate in sign. This result provides the easiest proof of the existence of nonalternating link types. Let L be an admissible link type with a connected, alternating, directed link projection P. Then there exists an effective algorithm, which when applied to P yields a link projection P' also of type L. The algorithm is intuitively a process of untwisting. Then: (iii) If the number of crossings of P' is d and the determinant of L is D, then $D \geqq d$. A corollary is: The group of an alternating knot type K is infinite cyclic if and only if K is trivial. Result (iii) was stated for $\mu = 1$ by Bankwitz (C. Bankwitz, *Ueber die Torsionzahlen der Alter nierenden Knoten*, Math. Ann. vol. 103 (1930) pp. 145–161). (Received October 24, 1955.)

383t. Eldon Dyer: *One-dimensional open maps onto Cantorian manifolds.*

In this paper it is shown that if f is a monotone open (continuous) mapping of a compact metric space X onto an n-dimensional Cantorian manifold Y and for each point y of Y, $f^{-1}(y)$ is one-dimensional and lc^1 (with coefficient group the reals mod 1), then X is an $(n+1)$-dimensional Cantorian manifold. The proof relies on category arguments and homological dimension theory. Also use is made of an earlier result of the author announced in an abstract entitled *Open mappings and regular convergence* (Bull. Amer. Math. Soc. Abstract 62-2-275). This theorem generalizes certain results of Hamstrom and Anderson (Proc. Amer. Math. Soc. vol. 6 (1955) pp. 766–769) and strengthens a result of the author (to appear soon in Ann. of Math.). (Received January 12, 1956.)

384. Eldon Dyer: *Open mappings and dimension.* Preliminary Report.

The following result is established in this paper. Suppose f is an open (continuous) map of a compact metric space X onto an n-dimensional space Y and that for each point y of Y, $f^{-1}(y)$ is homologically locally connected (with coefficient group the reals mod 1) in all dimensions. Suppose further that there are integers p and q, $p<q$, such that for each point y of Y, $p \leqq \dim (f^{-1}(y)) \leqq q$. For each integer i, let Y_i denote the set of all points y of Y such that $\dim (f^{-1}(y)) = i$. Then there is a greatest integer r such that Y_r intersects every dense G_δ subset of Y, and $r+n \leqq \dim (X) \leqq q+n$. In case $\dim (f^{-1}(y)) = q$ for each point y of Y, $\dim (X) = q+n$. The arguments depend on results announced by the author in an abstract entitled *Open mappings and regular convergence* (Bull. Amer. Math. Soc. Abstract 62-2-275) and use homological dimension theory. (Received January 12, 1956.)

385t. Herbert Federer: *A theorem on principal fibre maps.*

Suppose $f: E \to B$ is a fibre map in the sense of Serre, F is an (arcwise connected) fibre of f, and $g: E \times F \to E$ is a continuous, fibre preserving, associative multiplication with a unit $1 \in F$. (If $x \in E$, $y \in F$, $z \in F$, then $f(g(x, y)) = f(x)$, $g(g(x, y), z) = g(x, g(y, z))$, $g(x, 1) = x$, $g(1, y) = y$.) The corresponding Pontrjagin product makes the cubical singular complex M of E (using cubes with vertices at 1) a differential graded right module over the Pontrjagin cubical chain ring A of F, in the sense of H. Cartan (École Normale Supérieure, Séminaire 1954/1955). Let I be the kernel of the usual augmentation of A and consider Cartan's associated quotient module $\overline{M} = M/MI$. It is proved that *the natural homomorphism induced by f maps the homology groups of $\overline{M}$ isomorphically onto the cubical singular homology groups of B.* (The proof uses a general theorem of J. C. Moore to compare two filtrations of M, namely the fibre space filtration of Serre and the differential graded module filtration of Cartan.) One consequence of this result is a simplification of the homology theory of spaces of type $K(\pi, n)$, because the geometric significance of the "bar construction" can now be established directly without use of the "construction W." (Received December 27, 1955.)

386. Jesús Gil de Lamadrid: *The bounded topologies in locally convex topological vector spaces.* Preliminary report.

Let E be a locally convex topological vector space and $\mathfrak{I}$ its topology. We denote by $\mathfrak{I}'$ the finest (not necessarily locally convex) topology which coincides with $\mathfrak{I}$ on bounded sets. We call $\mathfrak{I}'$ the bounded topology. The family of convex sets, open in $\mathfrak{I}'$, is a base of a locally convex topology $\mathfrak{I}''$, called the locally convex bounded topology. A mapping (linear mapping) of E into a topological space (into another locally convex space) is continuous in $\mathfrak{I}'$ ($\mathfrak{I}''$) if and only if it is continuous relative to—that is when restricted to—bounded sets. $\mathfrak{I}=\mathfrak{I}'$ ($\mathfrak{I}=\mathfrak{I}''$) if and only if mappings (linear map pings), continuous relative to bounded sets, are continuous. Examples of $\mathfrak{I}=\mathfrak{I}''$ are bornological spaces [Dieudonné, Bull. Amer. Math. Soc. vol. 59 (1953) pp. 495–512]. Among them are normed spaces, for which further $\mathfrak{I}=\mathfrak{I}'$. If $\mathfrak{I}=\mathfrak{I}''$ the space of continuous linear transformations of E into any locally convex space F is a closed (in the bounded-open topology) subspace of the space of all mappings of E into F. (Received November 28, 1955.)

387*t*. Jesús Gil de Lamadrid: *Spaces of mappings.* Preliminary report.

The terminology and notation of Abstract No. 387 is used. The purpose of this note is to establish certain topological results regarding subspaces of the space of all mappings of E into F. If this space of mappings is given the topology of uniform convergence over the bounded sets of E, the main results state that the following vector subspaces are closed: the space of bounded mappings (mapping bounded sets of E into bounded sets of F), the space of mappings that are continuous with respect to the bounded topology of E, and the space of compact mappings (the latter in case F is complete). For a closely related result see J. W. Brace, Bull. Amer. Math. Soc. vol. 60 (1954) p. 152. The proofs of these results are all similar. The case of compact mappings depends on the lemma: if A is a bounded subset of E and Φ is a maximal filter of A, then for every compact mapping f of E into F, $f(\Phi)$ is a convergent filter of F. From the above and other results it follows that a mapping is compact if and only if it is the uniform limit over bounded sets of bounded finite dimensional mappings (mapping bounded subsets of E into finite dimensional vector subspaces of F). (Received December 30, 1955.)

388. Friedrich Huckemann: *On the line-complex representation of Riemann surfaces.*

The representation of simply connected Riemann surfaces, which carry singularities only over a finite number of base points, by a line-complex, is well known [Speiser, R. Nevanlinna, Elfving] and widely used with success in problems within the theory of distributions of values. It seems desirable, therefore, to extend the notion of the line-complex as to cover a larger class of Riemann surfaces, in particular in order to deal with critical points of Riemann surfaces. Steps in this direction have been taken already, an axiomatic definition of a generalized notion of a line-complex, however, was still missing. A topological mapping now leads to the definition of a line-complex which permits to represent all those (simply connected) Riemann surfaces which have all their singularities over a closed Jordan curve without multiple points. A necessary and sufficient condition determines whether a line-complex does represent such a Riemann surface. A certain structure, called "boundary element," in a line-complex determines uniquely a critical point of the surface represented, while between different boundary elements a certain relation holds if and only if these elements determine the same critical point. (Received January 11, 1956.)

389. B. H. McCandless: *Test spaces for dimension n.*

If n is a non-negative integer, a separable metric space Y^n is called a *test space for dimension n* if it satisfies the statement: A separable metric space X has dimension $\leqq n$ if, and only if, given a closed subset C of X and mapping $f: C \rightarrow Y^n$, there is an extension of f over X. The purpose of this paper is to characterize test spaces for dimension n. The following result is obtained: Let n be a non-negative integer. A compact, n-dimensional n-LC space Y^n is a test space for dimension n if and only if (i) $\pi_i(Y^n)=0$, $i=0, 1, \cdots, n-1$. (ii) Y^n is LC^{n-1}, (iii) $\pi_1(Y^1)$ is infinite cyclic if $n=1$; if $n \geqq 2$, there exists an integer $k_n>0$ such that $\pi_n(Y^n)$ is the sum of k_n infinite cyclic groups. (Here $\pi_0(Y)=0$ means that Y is arc-wise connected.) Results due to Kuratowski and Hurewicz are utilized in the proof. (Received January 9, 1956.)

390. Paul Olum: *The second level in obstruction theory.*

Given f, $g: X \rightarrow Y$ with $f|A=g|A$, this paper defines effectively computable invariants at the "second level" of depth bearing on the homotopy of f and g rel. A. Let n and $q>n$ be integers, Π and G be abelian groups; take elements $y \in H^n(Y; \Pi)$, $w \in H^{q+1}(\Pi, n; G)$ such that $w \vdash y=0$, and $x \in H^{n-1}(X, A; \Pi)$; (if $n=1$, with appropriate modifications, Π may be non-abelian and operate on G). Assume the vanishing of the "first level" invariants, namely $(\dagger): (f-g)^*y=0$ in $H^n(X, A; \Pi)$ and $(f-g)^*: H^q(Y; G) \rightarrow H^q(X, A; G)$ has image 0. Following the ideas of Steenrod's functional operations, a function $\lambda_{f,g}(y, w, x)$ is defined purely homologically in terms of certain mapping cylinders. Denote by $\Lambda_{f,g}(y, w)$ the image set of $\lambda_{f,g}$ for fixed y, w. Then $f \simeq g$ rel. A implies that, for all (y, w) as above, $(\dagger)$ is satisfied and $0 \in \Lambda_{f,g}(y, w)$. For the second obstruction (i.e. $\pi_r(Y)=0$ for $r<n$ and $n<r<q$, $\pi_n(Y)=\Pi$, $\pi_q(Y)=G$), and (X, A) polyhedral of dimension q, the converse is true. Corresponding definitions and results hold for the extension problem. For $Y=n$-sphere, $X=(n+1)$-complex, the results reduce to those found earlier by Massey (unpublished). (Received January 12, 1956.)

391*t*. Paul Olum: *Classification of mappings of compact 2-manifolds.*

Let $f: M \rightarrow N$ map one compact connected 2-manifold into another, inducing $\theta_f: \pi_1(M) \rightarrow \pi_1(N)$ and having twisted degree $\deg f$ (cf. *Mappings of manifolds and the notion of degree*, Ann. of Math. vol. 58 (1953) pp. 458–480). Let P be the projective plane. Then θ_f and $\deg f$ are known to be a sufficient set of invariants for determining the homotopy class of f provided either that $N \neq P$ or that $N=P$ and θ_f is orientation-true (loc. cit. p. 464). The present paper solves this problem in the remaining case. Let f, $g: M \rightarrow P$ be order-preserving simplicial maps with θ_f, θ_g not orientation-true. Suppose now (1) $\theta_f=\theta_g$ and (2) $\deg f=\deg g$ (mod 2). Using integers mod 2 as coefficients throughout, let z^1 be a nontrivial 1-cocycle in P; there are then two 0-cochains c^0 and $c^{0\prime}$ such that $f^*z^1-g^*z^1=\delta c^0=\delta c^{0\prime}$. Define a 2-cocycle z^2 in M by $z^2=f^*z^1 \cdot f^*z^1 \cdot c^0+f^*z^1 \cdot c^0 \cdot g^*z^1+c^0 \cdot g^*z^1 \cdot g^*z^1$; similarly for $z^{2\prime}$ using $c^{0\prime}$. Define $\mu=\sum z^2(\sigma_2)$, summed (mod 2) over all 2-simplexes $\sigma_2 \in M$; similarly for μ'. Theorem: $f \simeq g$ if and only if (1) and (2) hold and $0 \in \{\mu, \mu'\}$. The proof is an application of the results announced in the preceding abstract. (Received January 12, 1956.)

392. J. P. Roth: *A function-space formulation of the switching-system problem.*

The actual physical problem facing the designer of switching systems is considerably deeper than its logical formulation (cf. abstract 367); it is given here the following topological formulation. A truth function ϕ defines a cubical complex K contained in an n-cube Q, also determined by ϕ. A *pair* (X, A) consists of a linear graph X and an *assignment* A which attaches to each branch (1-cell) of X an $(n-1)$-cube of Q. Let p, q be distinct 0-cells of X and α an (acyclic) geometrical 1-chain linking p and q; A defines a corresponding algebraic 1-chain α^*, the coefficient of a cell σ of 0^* being the $(n-1)$-cube assigned to σ by A. The *Kronecker index* of α^* is the set-theoretic product of its coefficients. Let $\Omega(X, p, q; A)$ be the space of all such α^*; the pair (X, A) shall be *admissible* if the union of the Kronecker indices of the elements of Ω is precisely the complex K. The *switching-system problem* is now to find an admissible pair such that the number of branches of X is a minimum. The multiple-terminal problem —a logical formulation of which is unknown to the writer—admits of a similar though more complex interpretation. The topological problem has in general a lower minimum than the logical problem. Routine algebraic methods are given for recognizing whether the bridge circuit is admissible. (Received January 10, 1956.)

393. Sol Schwartzman: *Asymptotic cycles.* I.

Let X be a locally connected compact metric space whose first Betti number is finite. Suppose the real line acts as a topological transformation group on X. Recall that $x_0 \in X$ is called quasi-regular if $\lim_{t\to\infty} (1/T)\int_0^T f(x_0 t)dt$ exists for every continuous $f(x)$. It is well known that the set of all points that are not quasi-regular has measure zero with respect to every finite invariant measure. Take any fixed quasi-regular point x_0. For each $t>0$, let K_t be an arc going from $x_0 t$ to x_0. We suppose the K_t chosen so they can be parametrized equi-uniformly continuously. The orbit from x_0 to $x_0 t$ followed by the arc K_t is a closed curve. Let C_t be the corresponding element of the first Betti group. Theorem: $\lim_{t\to\infty} C_t/t$ exists and is independent of the choice of the arcs K_T. Thus we have assigned to each quasi-regular x_0 an element of $B^1(X)$. If μ is any finite invariant measure we call the (μ)-average of this vector valued function the (μ)-asymptotic cycle C_μ. General formulas are obtained for computing C_μ. Applications to theorems about continuous eigenfunctions, cross sections in the large and related problems will be given at a later date. (Received January 12, 1956.)

394*t*. Sol Schwartzman: *Asymptotic cycles.* II.

Let X be a compact differentiable manifold of class C^2 and dimension $2M$. Assume given on X a closed nonsingular differentiable 2-form ω. An admissible coordinate system will mean a local C^2 coordinatization from a $2n$-dimensional sphere with coordinates p_i, q_i such that $w = \sum dp_i dq_i$ in this coordinate system. Finally let α be a differentiable closed one-form. In any admissible coordinate system, choose a function H such that $dH = \alpha$. Set up the Hamiltonian equations $dq_i/dt = \partial H/\partial p_i$, $dp_i/dt = -\partial H/\partial q_i$. Obviously these equations depend only on α and not on the choice of H. The restriction to admissible coordinates guarantees that a change of coordinate systems gives a compatible set of equations; so we obtain a one-parameter flow on X. By Liouville's theorem the exterior product of ω with itself n times gives an invariant measure μ. Theorem: The μ-asymptotic cycle for the flow defined by the one-form α

depends only on the cohomology class to which α belongs. The map which sends each cohomology class into the corresponding μ-asymptotic cycle is a homomorphism. In particular, if we insist on a single-valued Hamiltonian function, so that $\alpha = dH$, the μ asymptotic cycle is zero. (Received January 12, 1956.)

395. Stephen Smale: *A Vietoris mapping theorem for homotopy.*

Let X and Y be compact metric spaces and $f: X \to Y$ be onto. Vietoris proved (Math. Ann. (1927)) that if for all $r \leqq n-1$ and all $y \in Y$, $H_r(f^{-1}(y)) = 0$ (Vietoris homology mod 2) then $f_*: H_r(X) \to H_r(Y)$ is an isomorphism onto for $r \leqq n-1$ and onto for $r = n$. Begle (Ann. of Math. (1950)) gave generalizations to nonmetric spaces and more general coefficient groups. Examples show that this theorem does not hold directly for homotopy. However, suppose X and Y are LC^n and that for each $y \in Y$, (1) $f^{-1}(y)$ is LC^{n-1} and (2) $\pi_r(f^{-1}(y)) = 0$ for $0 \leqq r \leqq n-1$; then it is proved that $f_\#: \pi_k(X) \to \pi_k(Y)$ is an isomorphism onto for $k \leqq n-1$ and onto for $k = n$. Actually the hypotheses can be weakened in some respects. For example, if instead of assuming $\pi_{n-1}(f^{-1}(y)) = 0$, one only has that $i_\#: \pi_{n-1}(f^{-1}(y)) \to \pi_{n-1}(X)$, the homomorphism induced by inclusion, is zero, then still $f_\#: \pi_{n-1}(X) \to \pi_{n-1}(Y)$ is an isomorphism onto. (Received December 14, 1955.)

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RESEARCH PROBLEM

9. Richard Bellman. *Minimization problem.*

We are given a region R and a random point P within the region. Determine the paths which

(a) Minimize the expected time to reach the boundary, or

(b) Minimize the maximum time required to reach the boundary.

Consider, in particular, the cases

(a) R is the region between two parallel lines at a known distance d apart.

(b) R is the semi-infinite plane and we are given the distance d from the point P to the bounding line. (Received November 18, 1955.)