ASYMPTOTIC PHENOMENA IN MATHEMATICAL PHYSICS

K. O. FRIEDRICHS

The problems I intend to speak about belong to the somewhat un-defined and disputed region at the border between mathematics and physics. The fields of physics from which these problems originate are rather classical; the mathematical questions involved are also rather classical. That does not mean that these problems belong to the past. On the contrary, they are quite alive today and—I am convinced—they will remain so for some time.

The problems concern what may be called asymptotic phenomena. Instead of explaining in general terms what I mean by asymptotic phenomena, I prefer to single out at first one class of such phenomena: *discontinuities*. A typical discontinuity of the kind I have in mind is the boundary of the *shadow* which appears when a light wave passes an object. Now, the propagation of light is governed by a partial differential equation which has continuous solutions. How then is it possible that a discontinuity arises? Of course, actually there is no sharp discontinuity at the shadow boundary; there is a transition from light to dark which takes place across a very narrow strip along the shadow boundary. Nevertheless, it is remarkable enough that the differential equations of wave motion have solutions which involve such quick transitions—in fact, most differential equations of physics possess such solutions—and it is an interesting task to study those features of these equations which make such quick transitions pos-sible.

Discontinuities and quick transitions occur in various branches of physics. A striking example of a discontinuity is the *shock* in gas motion. Quick transitions occur frequently in situations in which one perhaps would not speak of a discontinuity. A case in point is Prandtl's ingenious conception of the *boundary layer*. This is a narrow layer along the surface of a body, traveling in a fluid, across which the flow velocity changes quickly. Prandtl's observation of this quick transition was the starting point for his theory of fluid resistance. Other cases, closely related to the boundary layer phenomenon, are the so-called *edge effect* in the deformation of elastic plates and shells and the *skin effect* in the flow of electric currents. A number of other such effects will be described in the later parts of this lecture. All

these effects may be regarded as typical asymptotic phenomena.

Without attempting to give a precise definition of this term, I shall simply call asymptotic all those phenomena which show discontinuities, quick transitions, nonuniformities, or other incongruities resulting from approximate description.

In the mathematical treatment of such phenomena, physicists have to a certain degree relied on their intuition, and very effectively so; but they have also developed or employed systematic mathematical procedures.

In such a systematic approach one may develop an appropriate quantity with respect to powers of a parameter, ϵ. This expansion is to be set up in such a way that the quantity is continuous for $\epsilon > 0$ but discontinuous for $\epsilon = 0$. Naturally, a series expansion with this character must have peculiar properties. A most remarkable property is that in general these series *do not converge*.

No doubt, divergent series are very useful; it has even been said that they are more useful than convergent ones. However this may be, if a divergent series is useful it must be meaningful.

The use of a series which does not necessarily converge is a typical instance of a "formal procedure" and I should perhaps say a word about the role of *formal procedures* in mathematical physics.

Those who employ mathematics as a tool have rarely been inhibited by the fear of divergence; they have always been confident that, somehow or other, formal procedures are valid. A mathematician may be inclined to frown on this attitude as a superstition; but, on second thought he will yield and try to show that formal procedures—I mean those used by good physicists—indeed are valid if only the meaning of validity is properly interpreted.

There are numerous instances of justification of formal procedures by re-interpretation. I need only refer to the generalizations of the notions of function and differential operator, which have been very effective, in particular, in recent years. It would certainly be interesting to trace the effect of these generalizations in mathematical physics; but I do not intend to do so.

The present talk will be solely concerned with the formal expansions used in the analysis of asymptotic phenomena. The idea of giving validity to these formal series is classical: essentially it goes back to Poincaré.

Poincaré advanced this idea in his work on ordinary differential equations in 1886. Before that time many formal series solutions of such equations had been developed and it was found that they did not converge—in general. Poincaré proved that these formal series

solutions represent *asymptotic expansions of actual solutions.* Thus it became clear in which way formal series solutions may be regarded as "valid."

Let me explain the meaning of the phrase "formal series solution" and its "asymptotic" character in connection with an elementary differential equation, namely the *differential equation of the second order,*

$$(1) \qquad\qquad \epsilon u'' + au' + bu = 0.$$

Here u is a function of a variable z, which in the present context may just as well be taken as real. Furthermore, a and b are analytic functions of z and ϵ is a parameter. Note that this parameter occurs in such a way that the order of the differential equation is reduced for $\epsilon = 0$.

We are interested not in solutions of this differential equation for each fixed value of the parameter ϵ, but in the dependence of such solutions on this parameter, in particular, in the neighborhood of $\epsilon = 0$.

The formal series which we shall consider are not simply power series in ϵ; they are rather of the form

$$(2) \qquad\qquad e^{S(z)/\epsilon} \sum_n \epsilon^n v_n(z),$$

which will be referred to as the "standard" form. One may try to find solutions of the differential equation (1) which admits such a series expansion. To this end one tries to determine the functions S and v_n by inserting this series into the differential equation and setting the coefficient of every power of ϵ equal to zero. For the functions S and v_n one then finds simple differential equations which are easily solved. The equation to be satisfied by the function S, the so-called "characteristic equation," is

$$(3) \qquad\qquad (S')^2 + aS' = 0.$$

Here S' is the derivative of S. Inserting the functions S and v_n thus found into the series (2) one obtains a *formal series solution* of the differential equation. In general, though, this series does not converge.

A formal series of the type described is said to represent the *asymptotic expansion* of a function $u(z, \epsilon)$ if the remainder of the terms up to the Nth order is of the order $N+1$; precisely, if

$$e^{-S(z)/\epsilon} u(z, \epsilon) = \sum_{n=0}^{N} \epsilon^n v_n(z) + O(\epsilon^{N+1}),$$

uniformly in an appropriate z-interval.

The problem treated by Poincaré was a little different from the one just described since he did not consider expansions with respect to powers of ϵ but with respect to powers of z^{-1}. Nevertheless, it was to be expected that the analogue of what he proved also holds in the case considered here. That is, each formal series of the standard type should be the asymptotic expansion of an actual solution. That this is so for equations of the second order was proved as early as 1899 by Horn. The corresponding general theory for equations of the nth order, developed in 1908 by Birkhoff, initiated an extensive literature in this and related fields.

These results may be used to answer questions concerning the behavior of specific solutions of the differential equation as the parameter ϵ tends to zero. I should like to discuss one such question, which is extremely elementary, but nevertheless leads in a natural way to the boundary layer phenomenon.

Let us *prescribe boundary values* for the solution of our differential equation at two points, $z = 0$ and $z = z_1$, and ask how the solution of this boundary value problem behaves as $\epsilon \to 0$. Note that for $\epsilon = 0$ the differential equation reduces to an equation of the first order. One may therefore wonder whether the solution of the equation of the second order approaches a solution of the equation of the first order. Now a solution of the first order equation is already determined by one boundary condition; one cannot expect that both conditions will be satisfied in the limit. One *boundary condition*—at least—*will get lost*. The question is, which one?

This question and related questions can easily be answered with the aid of two solutions possessing a standard expansion. The answer is that under appropriate conditions the solution of the boundary value indeed does converge to a solution of the first order equation. This solution assumes one of the two boundary values but not the other one. Which boundary value is lost depends on the sign of a/ϵ. Let us assume that the lost boundary value is the one prescribed at $z = 0$.

The process of losing a boundary value takes place through *non-uniform convergence*. If the parameter ϵ is small, the solution will run near the limit solution except in a small segment at the end point $z = 0$ where it changes quickly in order, as it were, to retrieve the boundary value about to be lost.

Thus a "quick transition" is found to occur. It must occur since a boundary condition is about to be lost; and this loss in turn is necessary since the order of the differential equation is about to drop. To be

sure, the *reduction of the order of a differential equation* combined with the *loss of a condition such as a boundary condition* is the most characteristic mathematical feature of asymptotic phenomena.

The next step in the asymptotic analysis of our problem consists in a detailed *description* of the solution of the boundary value problem *within the transition layer*. To this end we introduce a new independent variable by *stretching* the original variable in an appropriate manner. Specifically, we introduce the ratio

$$\zeta = z/\epsilon$$

as a new variable. We then consider the quantity u as a function of ζ, in addition to ϵ, and ask whether or not this new function approaches a limit as $\epsilon \rightarrow 0$. This is indeed the case. The limit process is now uniform even near $z = 0$. Therefore the new limit function may serve as an approximate description of the quick change of u in the transition layer.

The new limit function is defined for all $\zeta \geq 0$; in fact it approaches a definite value as $\zeta \rightarrow \infty$. Remarkably enough, this value of u at $\zeta = \infty$ is exactly equal to the value which the limit function in the first "direct" process assumes at $z = 0$.

This peculiar phenomenon may at first sight appear a little paradoxical, but actually, it is quite natural. Evidently, any fixed z-neighborhood of the point $z = 0$ corresponds to an arbitrarily large part of the ζ-axis if only ϵ is made sufficiently small. It is therefore clear that a connection of the two limit functions must involve the behavior of the direct limit function at $z = 0$ and the behavior of the second limit function at infinity. The phenomenon just described will be referred to as "identification phenomenon."

The results discussed in connection with the simple equation of the second order are rather typical and they may frequently serve as a guide in understanding other asymptotic phenomena.

As an example, let us consider Prandtl's *boundary layer theory*. This theory was developed in order to solve the problem of fluid resistance, which had caused great difficulties since the time of d'Alembert. It was known that the resistance is due to the viscosity of the fluid; for, it was known that nonviscous fluids do not exert a force on bodies through it. Still, for fluids with low viscosity the assumption of absence of viscosity led to a very satisfactory description of the flow around the body, although it did not yield a resistance.

Prandtl in 1904 resolved this dilemma by advancing the hypothesis that the effect of viscosity is concentrated in a narrow layer near the

surface of the body. On the basis of this hypothesis he was able to give a detailed description of the flow in it. With unfailing intuition he appraised the order of magnitude of the various terms of the governing differential equation and rejected those that he judged to be insignificant. The simplified equations thus obtained could then be solved.

There was never any doubt that the boundary layer theory gave a proper account of physical reality, but its mathematical aspects remained a puzzle for some time. Only when this theory is fitted into the framework of asymptotic analysis, does its mathematical structure become transparent.

Viscous fluid flow—in two dimensions, for simplicity—is governed by a partial differential equation of the fourth order. If the viscosity vanishes, the equation reduces to one of the third order. The expansion of viscous fluid flow in the neighborhood of inviscid fluid flow thus appears as an asymptotic expansion, the viscosity being the parameter. A viscous fluid sticks to the wall; hence two boundary conditions are imposed on the viscous fluid flow: namely the conditions that the tangential and normal velocity components vanish. An inviscid fluid is permitted to slide; hence only one condition is imposed on it. Thus one boundary condition gets lost when the viscosity becomes zero.

It is now clear that, before the boundary condition is lost, a quick change must take place across a thin layer near the boundary. This layer, of course, is Prandtl's boundary layer.

Prandtl's detailed description of the flow in the boundary layer can be re-derived by a stretching procedure similar to the one described above. The new stretched variable must be so chosen that with respect to it the boundary layer does not shrink to zero as the viscosity tends to zero.

The approach to the boundary layer theory outlined leads to a definite clarification of the issue but it does not yield a rigorous justification of this theory. The main reason for this difficulty is the nonlinearity of the problem.

The situation is similar in many other nonlinear asymptotic problems. Methods for approximate solutions of such problems are frequently suggested by the facts discussed in connection with the simple ordinary differential equation of the second order.

The boundary layer effect discussed is not the only form of breakdown of uniform convergence. Such a breakdown may also happen in the interior of the domain. A most remarkable such occurrence, in its mathematical aspects even more striking than the boundary layer

phenomenon, is the phenomenon discovered by *Stokes* in 1857.

This phenomenon may be explained in connection with the simple differential equation of the second order (1) discussed before. In doing this it is preferable to assume that the coefficients and the solution are analytic functions of the complex variable z.

It was mentioned above that every formal series solution of the standard type is the asymptotic expansion of an actual solution of equation (1), but it was not stated where this asymptotic expansion is valid. In fact, it may happen that this expansion is valid in only a part of the domain in which the function $u(z, \epsilon)$ is defined. That is, it may happen that the function u admits the standard asymptotic expansion in only a part of the z-plane. In other parts it then will possess a completely different asymptotic expansion. The lines which separate subregions of different expansions are called "Stokes lines." The change of the asymptotic expansion on crossing these lines is the "Stokes phenomenon."

In short one may say: *the Stokes phenomenon* obtains at a line *if the asymptotic expansion of the analytic continuation* of u across this line *is not given by the analytic continuation* of the terms *of the asymptotic expansion.*

If a Stokes phenomenon is present, the leading term of the asymptotic expansion of the solution u changes its character on crossing the Stokes line; one may therefore say that this term is discontinuous across the Stokes line. Suppose the function u stands for a physical quantity and suppose this quantity is approximately described by the leading term of the expansion. If this term is discontinuous, the physical quantity is approximately described as being discontinuous, although actually it is continuous. Thus we have encountered the possibility of describing continuous quantities as discontinuous ones by describing them asymptotically. This possibility is of great significance. To be sure, a large class of *discontinuity phenomena in mathematical physics may be interpreted as Stokes or boundary layer phenomena.*

For an ordinary linear differential equation it is easy to locate lines at which a Stokes phenomenon occurs. One can always find such lines near a "turning point." A *turning point* or *transition point* is a point z at which two roots $S'(z)$ of the characteristic equation (3) coalesce: $S_1'(z) = S_2'(z)$. Stokes phenomena then may occur at certain rays through the turning point. For equations of the type here considered, these rays are curves on which the real parts of $S_1'(z)$ and $S_2'(z)$ agree: $\operatorname{Re} S_1'(z) = \operatorname{Re} S_2'(z)$.

For the differential equation (1), in particular, there are four such

rays issuing from a turning point. At which ones of these rays the Stokes phenomenon occurs depends on the solution considered.

Specifically, one can find a solution which possesses a standard asymptotic expansion in the open region R generated by three of the four sectors that are formed by the four rays. It possesses an asymptotic expansion also in the fourth sector, at the two rays bounding the fourth sector, and at the turning point; but all three expansions differ from each other and from the standard expansion in the region R. Clearly, a Stokes phenomenon is present; the two rays bounding the fourth sector are Stokes lines.

Of course, one wants to know how to find these different expansions.

Before indicating how one may attack this "continuation" problem I should mention that such a turning point problem was first treated by Jeffreys in 1923 in connection with a somewhat different differential equation.

One possible approach to a solution of the continuation problem consists in reducing it to the problem of finding the asymptotic expansion *at* the turning point. This problem will be referred to as the "connection problem."

I shall briefly indicate a formal procedure by which this connection problem may be attacked.

In this approach one again employs the "method of stretching." For equation (1), in particular, one introduces

$$\zeta = z/\epsilon^{1/2},$$

instead of z, as new independent variable, assuming the turning point to be at $z = 0$. It would not be difficult to motivate the choice of $\epsilon^{1/2}$ as stretching factor instead of ϵ. Again, any fixed neighborhood of the turning point will eventually cover the whole ζ-plane. The quantity u, when considered as a function of ζ, now possesses an expansion with respect to powers of $\epsilon^{1/2}$. The terms of this expansion, defined in the whole ζ-plane, can be determined by identifying their behavior at $\zeta = \infty$ with the behavior of the terms of the direct expansion at $z = 0$. In other words, one may employ an identification procedure similar to the one discussed in connection with the boundary value problem. The expansion of the solution u at the turning point can then be found.

As mentioned before, this approach is only a formal procedure; naturally, one will ask: does it yield correct results? For differential equation (1), in particular, it is not too difficult to prove that this is so. Such a proof is not so easy, however, in more complicated cases;

for example if an additional singularity is present, or if the equation is of higher order.

The first decisive step in developing a rigorous turning point theory was taken by Langer in 1934. Langer, in fact, treated at first a problem which is not as simple as the one discussed here. Subsequently, a considerable amount of work on the turning point problem has been done, and is being done today.

There are many interesting problems of *mathematical physics* in which a *turning point* analysis plays a role.

Wentzel, Brillouin, and Kramers in 1926 used asymptotic approximations and turning point considerations in solving eigenvalue problems in quantum theory.

A very remarkable problem which requires a turning point analysis is the problem of the *stability of viscous fluid* and the onset of turbulence. Quite a number of aerodynamicists and mathematicians have worked on this somewhat controversial question. Early theoretical investigations led to the prediction that such an instability should occur under peculiar circumstances. This prediction should perhaps have been believed by aerodynamicists, but it was not generally accepted at first. Eventually, the prediction was confirmed by experiment with surprising accuracy.

The pertinent mathematical situation was definitely clarified only in recent years by Wasow through a rigorous turning point analysis.

The asymptotic phenomena of ordinary differential equations which I have described up to now involve linear equations; of course such phenomena have also been studied in connection with *nonlinear* equations. An interesting problem concerns periodic solutions of a differential equation of the form

$$\epsilon u'' = f(u', u).$$

The question is what happens with these periodic solutions as $\epsilon \to 0$, in particular if the limit equation

$$f(u', u) = 0$$

has no periodic solution. Of course there could be no boundary layer effect in the strict sense since there is no boundary. What happens is that the limit function—if it exists—satisfies the equation $f(u', u) = 0$ except at certain points where the derivative u' has a jump discontinuity.

A problem of this type was first treated by van der Pol, 1927, who in this way explained the occurrence of certain jerky oscillations in electric networks, which he called "relaxation oscillations." Subse-

quently much work was done on electrical and mechanical oscillations of this kind, as well as on the purely mathematical aspects of the problem. Strong results on asymptotic periodic solutions have been obtained by Levinson since 1942.

Another, rather spectacular, case of a discontinuity which may take place in the interior of the domain and not at a boundary is the gas dynamical *shock*. The shock may also be interpreted as the limit of a quick transition and be treated by an asymptotic analysis which involves the drop of the order of a differential equation. The same may be said about the closely related phenomena of *explosion* or *detonation*.

Let me turn to *partial differential equations* and first consider the *hyperbolic* equation

$$(4) \qquad\qquad u_{tt} - \Delta u = 0,$$

called the "wave equation." Here u is a function of t, x, y, z and Δ is the Laplacian.

The propagation of electromagnetic and acoustic waves is governed by this equation; but these processes are frequently treated in a different manner, in the manner of geometrical optics. One is led to this second treatment in a natural way by asking for formal solutions of the "standard" form

$$(5) \qquad\qquad u = e^{S/\epsilon} \sum_n \epsilon^n v_n.$$

Here S and v_n are functions of t, x, y, z.

For these functions simple equations are found and readily solved. These equations are of the first order; the drop in order, so typical for asymptotic problems, is thus apparent. The equation for S, the characteristic equation

$$(6) \qquad\qquad S_t^2 = (\nabla S)^2,$$

is nonlinear.

The formal series (5) is similar to that used for ordinary differential equations. There is a slight difference, however, since the parameter ϵ entering it does not occur in the differential equation. If a concrete problem is to be solved by using this formal solution, the parameter will have to be identified with one of the data of the problem, such as a wave length or a pulse width.

In accordance with the principle of Poincaré, one expects that there exist actual solutions having these formal solutions as asymptotic expansions. That this is so has apparently not yet been proved. One

should think, however, that the available methods of proving the existence of solutions of hyperbolic equations would be strong enough for this purpose. But, the primary interest of the asymptotic theory of the wave equation seems to lie in the asymptotic expansion of the solutions of specific problems.

In many cases one can describe a wave process with the aid of the leading term of the above expansion (5). This description now leads to *geometrical optics*. The function S is the eiconal, and the equation $S = $ const. describes the motion of a wave front. It has been known for a long time that the transition of wave optics to geometrical optics involves asymptotic expansion; but little attention was paid to the fact that this expansion enables one also to determine the propagation of the amplitude. A systematic exploitation of this possibility was started only about ten years ago by Luneburg.

The most interesting asymptotic phenomenon of wave motion occurs when the eiconal S develops singularities on certain surfaces, called *caustics*. Such a singularity may occur since the differential equation satisfied by S is nonlinear.

Suppose now the wave function u possesses a standard asymptotic expansion on one side of the caustic, then its expansion on the other side will be a different one. In other words, a *Stokes phenomenon appears at the caustic*.

The situation is similar at a *shadow boundary*. The *transition from light to shadow is also a Stokes phenomenon*. For, a shadow boundary is just a line across which the asymptotic expansion changes, in other words, a Stokes line. To determine the asymptotic expansion in the shadow region is an interesting problem which has not yet been solved completely.

In connection with the interpretation of the shadow as a Stokes phenomenon I may perhaps make a general remark about *the role of discontinuities in the description of nature*. On the one hand, discontinuities appear to play a secondary role, namely when they are considered as approximate descriptions of continuous phenomena involving quick transitions. On the other hand, discontinuities play a primary role. For, the experimental description of nature and the theoretical description based on it involves objects with more or less sharp outlines. Therefore, nature could not be described in this way if natural objects did not possess sharp outlines, i.e. discontinuities. In other words, the quantities employed to describe nature could not even be defined if discontinuities did not occur. In this sense, discontinuities appear to play a primary role.

It may be debated whether or not this situation involves a vicious

or a nonvicious circle. In any case, one may say that asymptotic description is not just a matter of imperfection, but is an essential element in the mathematical description of nature.

The next subject for discussion naturally would be *elliptic partial differential equations*. Various important mathematical results have been obtained in this field. I need only refer to the classical asymptotic theory of eigenvalues developed by Weyl, Courant, and Carleman.

Instead of discussing these results I prefer to discuss a few problems from *mechanics* which involve elliptic equations.

Let us first turn to problems of *elasticity*. One such problem arises when a thin circular disk is subjected to lateral pressure applied along the edge. The disk will deflect if this pressure is large enough. This problem was investigated by Stoker and myself in 1940. The main question was what happens if the lateral pressure is increased indefinitely or—what is equivalent—if one lets the thickness of the plate shrink to zero. The answer was quite unexpected to us.

The deflection w and the stress function ϕ, considered as functions of x and y, satisfy a pair of differential equations

$$h^2\Delta^2 w = f, \qquad \Delta^2\phi = g$$

in which Δ^2 is the biharmonic operator, and f and g are quadratic functions in the second derivatives of w and ϕ; furthermore, h is the thickness of the plate.

The equations which result when one sets $h=0$ imply a constant distribution of the pressure in the plate. The question then arose, what is the value of this pressure? Is it the value prescribed at the edge?

One really had no right to expect this, since the order of the system of equations drops if one sets $h=0$, and one must face the possibility that at least one boundary condition gets lost. This might be the boundary condition concerning the pressure. If so, there should be a thin boundary layer across which internal and external pressure are connected.

The answer to this question could be derived from a boundary layer analysis of the type described before with the aid of the method of stretching. The answer was that the interior limit pressure indeed is not equal to the external pressure; but in addition it was found that this pressure is negative, that is represents *tension*. Thus tension should prevail over most of the plate in spite of the fact that a compression is applied at its edge.

This result was very surprising and we wondered whether we were

not misled by having employed a boundary layer analysis heuris-
tically. However, the validity of this procedure was proved rigorously
in this case; in fact, the present case is one of the few involving non-
linear equations in which this was possible.

Quick transitions of stresses and strains at the boundary of a de-
formed elastic body have been observed in many cases. Instead of a
boundary layer phenomenon one then speaks of an "edge effect." A
number of edge effects closely related to the one discussed have been
treated in the last ten years. The first detailed mathematical analysis
of an edge *effect* was given by H. Reissner in 1912 in his theory of
shells.

A rather famous edge effect was observed much earlier. This is the
effect in the *bending of thin plates with free edges*. The differential
equation for the deflection w of such a plate is the biharmonic equa-
tion $\Delta^2 w = 0$. Two boundary conditions should be imposed on w on
mathematical grounds, but three conditions were strongly favored on
physical grounds. A pair of two very peculiar boundary conditions
were proposed by Kirchhoff in 1850 and later on justified by Kelvin
and Tait by qualitative arguments which essentially involved a
boundary layer. But only recently was this problem treated by a
consistent asymptotic analysis.

In such an analysis, all significant quantities must first be de-
veloped with respect to the thickness of the plate. A stretching tech-
nique of the type discussed earlier leads to a description of the stresses
in the boundary layer. In this way one finds that Kirchhoff's condi-
tions indeed are correct, but, in addition, one can clearly understand
in detail how the third boundary condition gets lost.

Asymptotic approximation of quantities defined in thin layers may
lead to strange phenomena. I should like to mention one such phe-
nomenon in connection with a problem in *fluid dynamics;* namely the
problem of determining the flow of a layer of fluid over a bottom
surface under the influence of gravity. This flow is described by a
potential function.

There exists a very effective approximate treatment of such flow
based on the assumption that the layer of fluid is very thin. This is the
so-called "shallow water theory."

A peculiar feature of this approximation is that in it the motion is
governed by a hyperbolic differential equation, while originally it is
described by a potential function, the solution of an elliptic equation.
Small disturbances, which in the original description would affect
the whole flow instantaneously would be propagated with a finite
speed according to the *shallow water theory*. How is this possible?

This discrepancy is a typical symptom of asymptotic approxima-
tion. The shallow water theory results from the leading term of an
expansion with respect to the average thickness of the layer. In
deriving this expansion one must stretch the vertical variable y and
keep the horizontal variable x. The potential equation then goes over
into the equation

$$h^2\phi_{xx} + \phi_{\eta\eta} = 0$$

where $\eta = y/h$. Clearly, for $h = 0$ the elliptic character of the differ-
ential equation is lost. One may perhaps hesitate to destroy the
potential equation and to spoil the advantage of working with po-
tential functions. Still, it is appropriate to do so. It has been sug-
gested to call this procedure the "method of spoiling."

Incidentally, it was recently shown by Hyers and myself that a
similar *method of spoiling* is the clue to a rigorous treatment of the
"solitary wave," i.e. a steady shallow water wave with a single hump.

One more problem from fluid dynamics should be discussed,
Prandtl's theory of the *airfoil of finite span*. If a thin wing of finite
span travels through the air, a vortex sheet will develop at the trailing
edge. A precise treatment of the resulting airflow offers insurmount-
able difficulties, but Prandtl gave an approximate treatment derived
from rather intuitive arguments. He replaced the wing by a line,
called "lifting" line at which the flow is assumed to have an ap-
propriate singularity and made other simplifications. This procedure
was strikingly effective in the case of normal flight, i.e. flight in the
direction perpendicular to the wing, but the method breaks down
when applied to a wing in yaw or a swept back wing.

This difficulty was quite recently overcome by a systematic asymp-
totic treatment of the problem. The wing was imbedded in a set of
wings with the same span and similar cross-section profiles. When the
chord ϵ of the profile approaches zero, the wing shrinks to a line, the
"lifting line." The potential function describing the airflow past the
wing is now developed with respect to ϵ about $\epsilon = 0$. The leading term
in this expansion describes exactly Prandtl's approximate flow; it can
easily be given explicitly as soon as the circulation around the lifting
line is known.

In two dimensional airfoil theory the circulation can be deduced
from the shape of the profile by Kutta and Joukowski's theory. In the
present treatment the shape of the airfoil has disappeared in the
limit. To find the missing circulation this shape must be recovered.
That can be done by the method of stretching. The new, stretched
variables may be so chosen that the profile remains fixed; but then

the length of the span tends to infinity as $\epsilon \to 0$. With respect to the new variables the flow approaches a new limit. The new limit flow is essentially a two dimensional flow around an airfoil with infinite span. Such flow is well determined if its behavior at infinity is known. Now, infinity in the new variables corresponds to the neighborhood of the lifting line in the original variables. By identifying the behavior of the terms in the direct expansion at the lifting line with the behavior at infinity of the terms of the expansion after stretching one is able to find the missing circulation.

In this way, one can retrieve Prandtl's results for a wing flying in the head-on direction, and, in addition, one can treat *wings in yaw, and swept back wings*, which up to now appeared not to be amenable to an approach employing a lifting line.

In most problems discussed so far quite *similar methods of asymptotic analysis* were employed. These methods led to success in a number of cases, but still their scope is limited. That applies, in particular, to the simple method of stretching which played such a prominent part in our discussion.

Although the scope of the method of stretching is rather limited, the general idea of employing appropriate *transformations of the independent variable*, depending on the parameter, seems to be very fruitful. The importance of this idea, which occurs already in Poincaré's work, was strongly emphasized by Lighthill and various specifically adapted transformations were employed by him and others with remarkable success. One of the goals which may be attained in this way is uniformity. That is, one desires an approximation which is uniformly valid on the boundary and off the boundary in a problem of the boundary layer type, or on the Stokes line and off the Stokes line when a Stokes phenomenon is involved.

There are innumerable other asymptotic problems in mathematical physics more or less related to those discussed. An important field of such questions is concerned with physical processes which do or do not approach a *steady limit* as time goes on indefinitely. In fact, the occurrence of *stability and instability* may be regarded as an asymptotic phenomenon; and the decision between stability and instability therefore requires an asymptotic analysis.

It should not be forgotten that the foundations of *statistical mechanics* originated by Boltzmann and Gibbs abound with asymptotic problems of great significance and great difficulty.

It is not my intention to speak about these various fields. There is only one field of physics in which asymptotic problems occur to which I should like to refer: *quantum theory*.

It would have been tempting to speak about the *quantum theory of fields*. Here physicists have developed formal series expansions with great ingenuity. These series, to say the least, do not converge, and yet, to an amazing degree, they make sense physically. To be sure, to justify these formal procedures is a challenge to the mathematician.

However, I want to confine myself to discussing one or two problems in quantum theory which are well understood mathematically.

The first of these problems concerns the differential equation

$$\epsilon i \, \frac{d}{dz} \, \psi \; = \; H(z) \psi$$

in which ψ is an element of a *Hilbert space* which depends on the variable z and H is a self-adjoint operator, which is also assumed to depend on z. One is interested in the asymptotic behavior of a solution for small values of ϵ.

The equation is essentially of the same type as the ordinary differential equation considered before, the only difference being that the function ψ is an element of a Hilbert space. For this reason, a few technical difficulties must be overcome; but the idea of asymptotic analysis is exactly the same as for ordinary differential equations.

The interest in this problem arises in connection with the "adiabatic theorem" in quantum theory. The differential equation is the *Schrödinger equation* for the state ψ of the system if z/ϵ is regarded as the time. The operator $H(z)$ is then a Hamiltonian which varies "slowly" if ϵ is small. The *adiabatic theorem* now states: if the state ψ was an eigenstate of H originally, $t = 0$ say, it will remain approximately an eigenstate if t increases provided ϵ is small enough.

This theorem now results from the leading term of the asymptotic expansion of the solution ψ. In addition, however, the asymptotic analysis enables one to determine terms of higher order in the expansion and thus to estimate the "probability of nonadiabatic transition."

This is of particular interest in the case in which *two eigenvalues* of the operator H *coalesce* at some time during the process and the question has been asked what happens in such a case. Now, if one looks at the problem as an asymptotic problem, one need only realize that coalescence of eigenvalues corresponds to *a turning point*. A turning point analysis then gives a simple answer.

I should like to *summarize* some of the ideas presented. I have tried to show that a great number of asymptotic problems in mathematical physics have important features in common: in particular the drop of the order of the differential equation and the loss of a continuity or

boundary condition. I have tried to show that many of these problems can be attacked by similar methods. These methods involve asymptotic expansion, and the analysis of the regions of nonuniformity by stretching or adjustment of the independent variables combined with an appropriate identification procedure.

Furthermore, as I had mentioned in connection with the problem of the shadow, *asymptotic description* is not only a convenient tool in the mathematical analysis of nature, it *has a* more *fundamental significance.*

This fact is also apparent in the relationship between classical and quantum mechanics; a few words may be said about this relationship.

When wave mechanics was discovered it was immediately recognized that the relationship between wave mechanics and classical mechanics is essentially the same as that between wave optics and geometrical optics. That is to say, classical mechanics results from the leading term in the asymptotic expansion of quantum mechanics and in this sense classical mechanics plays a secondary role. On the other hand, as has been stated frequently in discussions of the foundations of quantum-theory, it is impossible to define and explain the basic notions of quantum mechanics without reference to classical mechanics. In this sense then, classical mechanics plays a primary role.

Thus we meet again the same circular situation which we had discussed in connection with the problem of the shadow. Indeed, *the relationship between classical and quantum mechanics affords a striking illustration of the fundamental role which asymptotic description plays in the mathematical description of nature.*

SELECTED REFERENCES

From the vast literature on asymptotic phenomena we shall present only a small selection. Additional references will be found in the publications quoted; but even with these additions the bibliography would be far from complete. No references will be given to work on various subjects which were only slightly touched upon in the lecture.

In particular, no reference will be made to the extensive literature on asymptotic series (e.g. the work of van der Corput) and on methods of asymptotic expansion of integrals and special functions (such as the saddle point method), except the résumé which Stokes gave on the phenomenon named after him,

G. G. Stokes, *On the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series,* Acta Math. vol. 26 (Abel Centenary Volume) (1902) pp. 393–397. Reprinted in *Mathematical and Physical Papers,* vol. V, pp. 283–287.

ORDINARY LINEAR DIFFERENTIAL EQUATIONS

Among the *classical work on asymptotic expansion* of solutions of ordinary differ-

ential equations we mention[1] that of Poincaré (1886), Horn (1899), Birkhoff (1908), Noaillon (1912), Tamarkin (1917), Perron (1918). Among more *recent work* we mention that of Trjitzinski (1934), Turittin (1936), and Hukuhara (1937). For the application of this theory to *eigenvalue problems* see the work of Tamarkin (1917 and 1927) and Birkhoff (1923); for the application to *boundary value problems* see Wasow (1944).

References to this work will be found in most *expository presentations* such as

G. D. Birkhoff and R. E. Langer, *The boundary problems and developments associated with a system of ordinary linear differential equations of the first order*, Proceedings of the American Academy of Arts and Sciences vol. 58 (1923) pp. 51–128. Reprinted in *Collected mathematical papers*, vol. I, pp. 347–424.

M. Hukuhara, *Sur les propriétés asymptotiques des solutions d'un système d'équations différentielles linéaires contenant un parametre*, Kyushu Imperial University, Faculty of Engineering Memoirs, vol. 8, 1936–1940, pp. 249–280.

H. L. Turittin, *Asymptotic expansions of solutions of systems of ordinary linear differential equations containing a parameter*, Contributions to the Theory of Nonlinear Oscillations, vol. II, Annals of Mathematical Studies, no. 29, Princeton University Press, 1952, pp. 81–116.

W. Wasow, *Introduction to the asymptotic theory of ordinary linear differential equations*, Report of the National Bureau of Standards, 1954, pp. 54–56.

For the work of Jeffreys (1923), Langer (1934), Cherry (1949), and others on the *turning point problem* see the

Review of the Literature, *Asymptotic solutions of differential equations with turning points*, Department of Mathematics, California Institute of Technology, Pasadena, 1953.

This is referred to as "Cal. Tech. Review" in the following.

NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

For a summary of earlier work see

M. H. Dulac, *Points singuliers des equations differentielles*, Memorial des Sciences Mathématique, no. 61, 1934.

For the later work by Hukuhara (1935), Nagumo (1939), Malmquist (1940) and others see

W. Wasow, *Singular perturbations of boundary value problems for nonlinear differential equations of the second order*, Communications on Pure and Applied Mathematics, vol. 9 (1956) no. 1.

For the work of Lighthill (1949) and others on *uniform expansions* see

H. S. Tsien, *The PLK method*, Advances in Applied Mechanics, vol. IV, Academic Press Inc., New York, 1955.

W. Wasow, *On the convergence of an approximation method of M. J. Lighthill*, Journal of Rational Mechanics and Analysis, to appear in 1955.

See also the work by Bromberg and Latta referred to below.

For the work by various authors on "singular perturbations" of solutions of *boundary or initial value problems* and of *periodic solutions* see

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[1] The year indicated frequently refers only to the first of a number of publications.

N. Levinson, *Perturbation of discontinuous solutions of nonlinear systems of differential equations*, Acta Math. vol. 82 (1950) pp. 71–106

and also

A. B. Vasilieva, Mat. Sbornik N.S. vol. 31 (1952) pp. 587–644.

RELAXATION OSCILLATIONS

For van der Pol's relaxation oscillations and the asymptotic theory of Haag (1943), Dorodnitsyn (1947), and others see

J. J. Stoker, *Nonlinear vibrations*, New York, Interscience, 1950, Chap. V. 6 and Appendix IV.

STABILITY

For a treatment of problems of stability see

R. Bellman, *Stability theory of differential equations*, New York, McGraw-Hill, 1953

and the survey article

V. V. Nemitzki, Uspehi Matematičeskih Nauk N.S. vol. 9 (1954) pp. 39–56.

PARTIAL DIFFERENTIAL EQUATIONS

The asymptotic theory of the equation

$$\epsilon\Delta u = Lu$$

in which Lu is an expression of the first order was treated by Wasow (1944) and

N. Levinson, Ann. of Math. vol. 51 (1950) pp. 428–445.

Asymptotic properties of the Navier-Stokes equation and related equations were investigated by Lagerstrom, Cole, Latta and others; see

P. A. Lagerstrom and J. D. Cole, *Examples illustrating expansion procedures for the Navier-Stokes equations*, Guggenheim Aeronautical Laboratory, California Institute of Technology, Pasadena, California, May, 1955.

WAVE EQUATION AND GEOMETRICAL OPTICS

R. K. Luneburg, Lecture Notes, Brown University, 1944, and New York University, 1947–1948.

F. G. Friedlander, Proc. Cambridge Philos. Soc. vol. 43 (1946) pp. 284–286.

F. G. Friedlander and J. B. Keller, *Asymptotic expansions of solutions of* $(\nabla^2 + k^2)u = 0$, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-67, September, 1954.

ELASTICITY

For the edge effect in shells and plates see

H. Reissner, *Spannungen in Kugelschalen (Kuppeln)*, Festschrift H. Mueller, Breslau, Leipzig, 1912, pp. 181–193.

O. Blumenthal, Archiv der Mathematik und Physik (3) vol. 19 (1912) pp. 136–174.

E. Reissner, Quarterly of Applied Mathematics vol. 10 (1952) pp. 167–173.

For the Kirchhoff's boundary conditions see a paper by R. Dressler and the author to appear in the Communications on Pure and Applied Mathematics.

For edge effects occurring in nonlinear problems see

K. O. Friedrichs and J. J. Stoker, Amer. J. Math. vol. 63 (1941) pp. 839–888,
E. Bromberg and J. J. Stoker, Quarterly of Applied Mathematics vol. 3 (1945) pp. 246–265

and a paper by E. Bromberg to appear in the Communications.

FLUID DYNAMICS

For the principles of *boundary layer theory* see

L. Prandtl, *The mechanics of viscous fluids,* Aerodynamic Theory, Ed. by W. F. Durand, vol. III G. (1935).
S. Goldstein, *Modern developments in fluid dynamics,* vol. I, Oxford, 1938.

For the *stability of viscous fluid flow* see the survey

W. Tollmien, *Laminare Grenzschichter,* Fiat Review of German Science, 1939–1946, vol. II, Hydro- and Aerodynamics, Wiesbaden, 1948, pp. 21–53.

Also see

C. C. Lin, *Hydrodynamic stability,* Proceedings of the Fifth Symposium in Applied Mathematics, New York, McGraw-Hill, 1954, pp. 1–18.

For the pertinent turning point analysis see the work by Wasow (1948–1952) quoted in the Cal. Tech. Review.
For the *shallow water theory* see the book

J. J. Stoker, *Water waves,* New York, Interscience

to appear.
For the principles of airfoil theory see the *Aerodynamic theory,* vol. I. For the approach described in the lecture see a paper by S. Ciolkowski and the author to appear in the Communications.

QUANTUM THEORY

For references to the work by Born and Fock (1926), Kato (1950) on the *adiabatic theorem* see a report by the author issued by the Institute of Mathematical Sciences, New York University.
The asymptotic expansion which leads from quantum mechanics to classical mechanics is clearly described by

G. D. Birkhoff, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 681–700.

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