Analytic functions. By S. Saks and A. Zygmund. Trans. by E. J. Scott. (Monografie Matematyczne, vol. 28.) Warsaw, Nakladem Polskiego Towarzystwa Matematycznego, 1952. 8+452 pp.

The theory of analytic functions has figured as a standard topic in the curriculum of a mathematics student for many years and there is a fairly clearcut agreement on what makes up the minimal contents of a first course on the theory of functions of a complex variable. However, when one examines the large number of texts on the subject, it is evident that standards of rigor and generality vary considerably. It is also clear that one can discern two quite distinct threads running through the fabric: first, the presence of arguments and methods which are very general—such as the use of topological notions (connectedness, compactness, interiority, etc.)—and are not peculiar to the theory of analytic functions; second, the presence of results and methods which are due to the particular features of the theory of analytic functions and give the theory its special color. In exposing a classical discipline, it is highly desirable that one should seek to isolate on the one hand the general tools and methods which are of constant use but are not peculiar to the discipline and on the other hand those features of the discipline which are special to it.

Such a program is envisaged by the book under review, the *Analytic functions* of Saks and Zygmund translated into English by E. J. Scott. Ever since its appearance in 1938, the original *Funkcje Analityczne* has evoked considerable interest far beyond the Polish mathematical public; tantalizing reference was frequently made to the manner in which certain topics were treated, a notable example being the elementary treatment of the Runge theorem concerning the approximation of analytic functions by polynomials and the exploitation of this result to prove the Cauchy integral theorem for simply-connected regions of the finite plane (i.e. regions having connected complement with respect to the extended plane) and to pave the way for the treatment of the Riemann mapping theorem.[1]

We learn from the authors' preface that it was their goal to take the middle road between a strictly "arithmetic" account of the theory along Weierstrassian lines and a "geometric" treatment which introduced certain intuitive geometric concepts without rendering them precise. "By no means renouncing the application of the

[1] We remark that the Runge approach was employed by Walsh in 1933 to prove with considerably more sophisticated apparatus (in fact, conformal mapping methods) the Cauchy-Goursat theorem for functions which are continuous in a closed Jordan region with rectifiable boundary and are analytic in the interior.

auxiliary apparatus of Geometry and the Theory of Sets, they tried to confine it to a domain in which it could be justified and made precise without undue difficulty for the beginner." This program has been brilliantly carried out and achieved without the sacrifice of any of the standard results that properly belong to a first course in the theory of analytic functions. Their approach was on the one hand to marshal up at the outset the general methods which are of constant application and to expound them in a general setting, and on the other hand to avoid appeal to the more delicate arguments of plane topology by employing adequate (for applications) elementary configurations. An example of the emphasis on elementary considerations is the proof of the Runge theorem cited above which is based on the use of square nets and the Cauchy integral formula for rectangles.

The Introduction is concerned with set theory, topological and metric spaces, elementary aspects of the topology of the plane. The first chapter deals with general questions of continuity, convergence, normal families, Ascoli's theorem, the elementary transcendental functions, and linear fractional transformations. From the outset exercises are introduced in abundance. Some are merely illustrative, others develop the theory further (e.g. Gauss' theorem on the zeros of the derivative of a polynomial, a very extensive and instructive set of problems on linear fractional transformations, Osgood's theorem on the pointwise limit of a sequence of analytic functions (in Chap. 2), etc.). The second chapter is concerned with the Cauchy theory treated in the elementary setting of systems of rectangles and the consequences of this restricted theory. The third chapter treats power series, meromorphic functions, isolated singularities, the Rouché and Hurwitz theorems, the interiority of meromorphic maps, and culminates in an introductory account of functions of two complex variables including the Weierstrass "Vorbereitungssatz." A dominant idea of the fourth chapter, *Elementary geometric methods*, is the possibility of approximating an analytic function by a rational function (which leads by the standard translation of pole technique to the Runge theorem in the simply-connected case) and the exploitation of this fact. The Cauchy theorem is now treated under more general hypotheses and residue theory is formulated with the aid of the notion of the order of a point with respect to a closed curve. The fifth chapter treats the Riemann mapping theorem and the Schwarz-Christoffel formulae. Chapter Six gives an excellent account of analytic continuation and puts to good use the abstract preliminaries of the introductory section. The use of meromorphic rather than analytic elements merits comment as does the proof of the monodromy theorem which is based on the Riemann mapping theorem. The selection of material from a rich field always involves a question of choice. An account of Riemann's classic work on the hypergeometric functions would have found a fitting niche in this chapter (cf. Ahlfors, *Complex analysis*) but its omission is certainly understandable.

The seventh chapter treats entire functions and meromorphic functions in the finite plane. The material treated includes the well-known expansion theorems of Weierstrass and Mittag-Leffler, growth questions, and the Picard theorems treated via the Bloch theorem. The remaining two chapters treat elliptic functions, the gamma and zeta functions, and Dirichlet series.

This brief account of the book indicates its scope and point of view. As we have remarked there is an abundance of exercises on which the good student may sharpen his mathematical teeth. He will have more than one occasion to test his skill with category arguments. On the other hand, the reader will note an absence of the treatment of the more delicate boundary problems which appeal either to a refined use of the topology of the plane or to methods involving Lebesgue integration. This is of course in accord with the stated program and intent of the book.

This book is a very welcome addition to the collection of texts on the theory of analytic functions which are now available in English. It will be a rewarding experience to the earnest student.

MAURICE HEINS

The stability of rotating liquid masses. By R. A. Lyttleton. Cambridge University Press, 1953. 10+150 pp. $6.50.

Ever since Newton deduced from his theory of gravitation that the shape of the earth must be an oblate spheroid, there has been intensive research into the question of the possible equilibrium shapes of rotating liquids. Maclaurin and Clairaut showed that for any value of angular momentum a spheroid is a possible equilibrium form. In 1834 Jacobi showed that if the angular momentum is greater than a certain amount an ellipsoid with three unequal axes is also a possible form of relative equilibrium.

The question of the stability of these equilibrium forms was first investigated by Poincaré in 1885. There are two different kinds of stability possible for rotating systems, known as "secular" and "ordinary" stability. To explain the distinction, consider a system rotating with constant angular velocity ω and assume it has n degrees of freedom $q=(q_1, q_2, \cdots, q_n)$ relative to a set of axes rotating with