THE CONVOLUTION TRANSFORM

D. V. WIDDER

Introduction. The material I am reporting on here was prepared in collaboration with I. I. Hirschman. It will presently appear in book form in the Princeton Mathematical Series. I wish also to refer at once to the researches of I. J. Schoenberg and his students. Their work has been closely related to ours and has supplemented it in certain respects. Let me call attention especially to an article of Schoenberg [5, p. 199] in this Bulletin where the whole field is outlined and the historical development is traced. In view of the existence of this paper I shall try to avoid any parallel development here. Let me take rather a heuristic point of view and concentrate chiefly on trying to entertain you with what seems to me a fascinating subject.

THE CONVOLUTION TRANSFORM

1. **Convolutions.** Perhaps the most familiar use of the operation of convolution occurs in its application to one-sided sequences $\{a_n\}_0^\infty$, $\{b_n\}_0^\infty$. The convolution (Faltung) of these two sequences is defined as the new sequence $\{c_n\}_0^\infty$,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k. \tag{1.1}$$

The operation arises when power series are multiplied together:

$$\sum_{k=0}^{\infty} a_k z^k \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} c_k z^k.$$

The convolution of two-sided sequences,

$$c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k} = \sum_{k=-\infty}^{\infty} a_{n-k} b_k, \tag{1.2}$$

presents itself when two Laurent series are multiplied.

Hardly less familiar is the continuous analogue of (1.2),

$$c(x) = \int_{-\infty}^{\infty} a(x-y) b(y) dy = \int_{-\infty}^{\infty} a(y) b(x-y) dy, \tag{1.3}$$

An address delivered before the Annual Meeting of the Society in Baltimore on December 29, 1953 by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings; received by the editors December 29, 1953.

which arises when two Fourier integrals or two bilateral Laplace integrals are multiplied together [6, p. 258]. It is customary to abbreviate the convolution operation by the symbol $*$, so that (1.3) becomes $c(x)=a(x)*b(x)$. If $a(x)$ and $b(x)$ both vanish on $(-\infty, 0)$, (1.3) reduces to

$$c(x) = \int_0^x a(x-y)b(y)dy, \tag{1.4}$$

the analogue of (1.1).

2. **The convolution transform.** We may interpret (1.3) as an integral transform, designating one of the functions $a(x)$ or $b(x)$ as the kernel, the other being then *transformed* into $c(x)$. It is small exaggeration to say that nearly all the integral transforms in mathematical literature are either in this form or can be put into it by change of variable. We give below a number of examples. In subsequent work we shall denote the kernel by $G(x)$ and suppose that $f(x)$ is the transform of $\phi(x)$:

$$f(x) = G(x)*\phi(x) = \int_{-\infty}^{\infty} G(x-y)\phi(y)dy. \tag{2.1}$$

EXAMPLE A. *A basic exponential transform.*

$$G(x) = g(x) = \begin{cases} e^x & (-\infty, 0), \\ 0 & (0, \infty), \end{cases} \tag{2.2}$$

$$f(x) = g(x)*\phi(x) = e^x \int_x^{\infty} e^{-y}\phi(y)dy. \tag{2.3}$$

EXAMPLE B. *The Laplace transform (unilateral).*

$$F(x) = \int_0^{\infty} e^{-xy}\Phi(y)dy. \tag{2.4}$$

Replace x by e^x, y by e^{-y}. Then (2.4) becomes

$$F(e^x)e^x = \int_{-\infty}^{\infty} e^{-e^{x-y}} e^{x-y}\Phi(e^{-y})dy,$$

and this is equation (2.1) if

$$f(x) = F(e^x)e^x, \qquad G(x) = e^{-e^x}e^x, \qquad \phi(x) = \Phi(e^{-x}).$$

EXAMPLE C. *The Stieltjes transform.*

$$F(x) = \int_0^{\infty} \frac{\Phi(y)}{x+y}dy.$$

This is in fact the first iterate of the Laplace transform (2.4). It takes the form (2.1) when

$$f(x) = F(e^x)e^{x/2}, \qquad G(x) = (1/2) \operatorname{sech} (x/2), \qquad \phi(x) = \Phi(e^x)e^{x/2}.$$

EXAMPLE D. *The Weierstrass (or Gauss) transform.*

$$f(x) = (4\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-y)^2/4}\phi(y)dy. \tag{2.5}$$

This is already in convolution form with kernel $G(x)$ equal to the Gauss frequency function $(4\pi)^{-1/2}e^{-x^2/4}$.

EXAMPLE E. *A general transform.*

$$F(x) = \int_0^{\infty} K(xy)\Phi(y)dy. \tag{2.6}$$

This is clearly equivalent to (2.1) after exponential change of variables. Many of the classical transforms appear in this form. Examples are: Laplace, Fourier-sine, Fourier cosine, Hankel, Meier. The five corresponding kernels are

$$K(x) = e^{-x}, \sin x, \cos x, J_0(x), x^{1/2}K_0(x),$$

where $J_0(x)$ is the Bessel function, $K_0(x)$ the modified Bessel function:

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos (x \cos y)dy,$$

$$K_0(x) = \int_0^{\infty} e^{-x \cosh y}dy.$$

We observe that the general Fourier, the bilateral Laplace, and the Mellin transforms can be expressed as the sum of two integrals (2.6).

In view of these examples the importance of the convolution transform as a unifying influence can scarcely be doubted. The two basic problems for any transform are (a) inversion, (b) representation. In (a) we seek to recover $\phi(x)$ from $f(x)$, the kernel $G(x)$ being known; in (b) we inquire what functions $f(x)$ can be written as convolutions (2.1) for a given kernel $G(x)$. In the present paper we restrict attention to the former problem, referring to [3] or [4] for the latter.

3. **Operational calculus.** A very useful guide to the study of the convolution transform is the operational calculus. Its practical importance was brought forcefully to public attention by Heaviside when he used it so advantageously in the study of electric circuits. In brief, the technique consists in treating some operational symbol,

such as D for differentiation, as if it were a number throughout some calculation and finally in restoring to it its original operational meaning. Of course the success of the method depends upon the existence of a correspondence between the operational laws of combination on the one hand, and the algebraic ones on the other, but there is obviously no compulsion to investigate this correspondence if one can check results directly (as in the case of Heaviside's differential equations).

Let us illustrate. Denote by D the operation of differentiation with respect to x and by a an arbitrary constant. From the symbolic expansion

$$e^{aD} = \sum_{k=0}^{\infty} \frac{a^k D^k}{k!}$$

we obtain the Maclaurin series

$$e^{aD}\phi(x) = \sum_{k=0}^{\infty} \frac{a^k \phi^{(k)}(x)}{k!} = \phi(x + a). \tag{3.1}$$

We now *define* $e^{aD}\phi(x)$ as $\phi(x+a)$ even if $\phi(x)$ is not differentiable. Suppose that we seek a solution of the differential equation

$$f(x) - f'(x) = \phi(x)$$

by the operational method. Using the symbol D, we have

$$\begin{gathered} (1 - D)f(x) = \phi(x), \\ f(x) = \frac{1}{1 - D}\phi(x), \end{gathered} \tag{3.2}$$

but it remains to interpret the operator $1/(1-D)$. Now

$$\frac{1}{1 - x} = \int_{-\infty}^{\infty} e^{-xy} g(y)dy, \qquad -\infty < x < 1, \tag{3.3}$$

where $g(y)$ is the function of Example A. This is easily verified by direct integration. Hence

$$\frac{1}{1 - D}\phi(x) = \int_{-\infty}^{\infty} e^{-yD}\phi(x)g(y)dy.$$

By (3.1) we thus obtain for the solution of (3.2)

$$f(x) = \int_{-\infty}^{\infty} \phi(x - y)g(y)dy = g(x) * \phi(x).$$

It is now an easy matter to verify that this is in fact a solution, at least for a large class of functions $\phi(x)$, by substituting the integral (2.3) in (3.2):

$$(1 - D)g(x) * \phi(x) = \phi(x).$$

Note that we have found an inversion operator, $1-D$, for the convolution transform (2.1) when the kernel $G(x)$ is the special function $g(x)$. Observe how it was obtained from equation (3.3).

More generally, this same operational method would lead us to expect that $E(D)$ would invert (2.1) if

$$\frac{1}{E(x)} = \int_{-\infty}^{\infty} e^{-xt} G(t)\,dt.$$

In other words, *the inversion function is the reciprocal of the Laplace transform of the kernel.* But how is $E(D)$ to be interpreted in the general case? If $E(x)$ is a polynomial there can be little doubt, but what if $E(x) = \cos x$ or $1/\Gamma(1-x)$, for example?

4. **The Laguerre-Pólya class.** We can make an effective interpretation of $E(D)$ if $E(x)$ belongs to a large class of functions originally studied by E. Laguerre (for references see Schoenberg [5]). We "normalize" the class, $E(0)=1$, in accordance with the following definition.

DEFINITION 4. *$E(x)$ belongs to class E if and only if*

$$E(x) = e^{-cx^2+bx} \prod_{k=1}^{\infty} \left(1 - \frac{x}{a_k}\right) e^{x/a_k}, \tag{4.1}$$

where a_k, b, c are real, $c \geqq 0$, and

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty.$$

For example, $1-x$, e^x, $\cos x$, $1/\Gamma(1-x)$, e^{-x^2} all belong to the class. Laguerre showed, Pólya introducing a refinement, that a function belongs to the class E if and only if it is the uniform limit of polynomials, each of which has real roots only and is equal to 1 at $x=0$. For example,

$$e^{-x^2} = \lim_{n\to\infty} \left(1 - \frac{x^2}{n}\right)^n.$$

Note that the corresponding equation for e^{x^2} would introduce polynomials with imaginary roots, and that $e^{x^2} \notin E$.

We mention this beautiful characterization of the class E only in passing, for we are not directly concerned with it here.[1] What does concern us is that the reciprocal of every function of E, except e^{bx}, is a Laplace transform, just as for $E(x)=1-x$ in equation (3.3). We state the result.

THEOREM 4.1. *If $E(x)\in E$ and $E(x)\neq e^{bx}$, then*

$$\frac{1}{E(x)}=\int_{-\infty}^{\infty}e^{-xy}G(y)dy, \tag{4.2}$$

the integral converging in the largest neighborhood of the origin which is free of zeros of $E(x)$.

It is not our purpose to give complete proofs of theorems here, but rather to outline methods. Let us first replace x by the complex variable $s=\sigma+i\tau$ and continue the function $E(x)$ analytically into the complex plane. We then show easily from (4.1) that $|E(\sigma+i\tau)| \geqq |E(\sigma)|$. Next, the product relation (4.1) may be used to show that

$$|E(\sigma+i\tau)|^{-1}=O(|\tau|^{-p}), \qquad |\tau|\to\infty,$$

uniformly in any vertical strip $|\sigma|\leqq R$. This is for any constant p, however large, if $E(s)$ is not the product of e^{bs} by a polynomial (in which case it is the degree of the polynomial). We can now appeal to a familiar theorem from the general theory of Laplace integrals [**1**, p. 126] which insures a representation (4.2) for any function $1/E(s)$ which is analytic in and uniformly small at the two ends of a vertical strip. Since $1/E(s)$ is analytic in the largest vertical strip containing the origin and free of zeros of $E(s)$ the representation (4.2) is valid there.

We need also some properties of the function $G(y)$ of (4.2).

THEOREM 4.2. *The function $G(y)$ of Theorem 4.1 has the properties:*

A. $$G(y)\geqq 0, \qquad -\infty<y<\infty,$$

B. $$\int_{-\infty}^{\infty}G(y)dy=1,$$

C. $$\int_{-\infty}^{\infty}yG(y)dy=b,$$

D. $$\int_{-\infty}^{\infty}(y-b)^2G(y)dy=c+\sum_{k=1}^{\infty}a_k^{-2}.$$

[1] In [**4**] we use it to show that the kernels $G(x)$ under discussion form a semi-group and hence in some sense exhaust *all* kernels amenable to our methods.

In the language of statistics, $G(y)$ is a frequency function of mean b and variance $c+\sum_{k=1}^{\infty} a_k^{-2}$. With the exception of A these properties are all corollaries of Theorem 4.1. Conclusion B follows from the normalization $E(0)=1$ when we set $x=0$ in (4.2); C follows by computing the first derivative of the integral (4.2) at $x=0$ and the computation is best done logarithmically; D follows by computing the second derivative of $e^{bx}/E(x)$ at $x=0$.

Property A is a little harder to prove, but the essential reason for its truth is almost intuitive. The reciprocal of each factor in (4.1) (except e^{bx}) is the Laplace transform of a non-negative function:

$$
\begin{aligned}
e^{cx^2} &= (4\pi c)^{-1/2}\int_{-\infty}^{\infty} e^{-xy}e^{-y^2/(4c)}dy, \\
\left(1-\frac{x}{a_k}\right)^{-1} e^{-x/a_k} &= |a_k|\int_{-\infty}^{\infty} e^{-xy}g(a_k y-1)dy,
\end{aligned} \tag{4.3}
$$

where $g(x)$ is the non-negative function of Example A. By the fundamental "product theorem" for Laplace transforms referred to in §1 we should expect that

$$
G(y) = (4\pi c)^{-1/2}e^{-(y-b)^2/4c} * |a_1| g(a_1 y - 1) * |a_2| g(a_2 y - 1) * \cdots
$$

if we are optimistic about matters of convergence. But then A is obvious since the convolution of positive functions is again positive.

In the more leisurely development available in a book [4] we have actually followed a different course in the proofs of both the theorems of the present section.

5. **Inversion.** Let us now show that the operator $E(D)$, if properly interpreted, is indeed effective for the inversion of (2.1). We treat first the case $c=0$, the factor e^{-cs^2} in (4.1) being then missing. For the purposes of presentation here we assume the simplest of conditions on $\phi(x)$. It should be emphasized, however, that the ultimate in generality has been achieved in this direction. We have shown that if $\phi(x)$ is any function for which (2.1) converges, then

$$
E(D)G(x)*\phi(x) = \phi(x)
$$

almost everywhere. This result should be contrasted with Jordan's theorem for Fourier series or integrals where some such local condition as bounded variation is needed.

THEOREM 5. *If $E(x)\in E$ with $b=c=0$, if $G(x)$ is the function of Theorem* 4.1, *and if $\phi(x)$ is bounded and continuous on* $(-\infty, \infty)$, *then*

$$
E(D)G(x)*\phi(x) = \phi(x), \qquad -\infty < x < \infty, \tag{5.1}
$$

where

$$E(D) = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \frac{D}{a_k}\right) e^{D/a_k}. \tag{5.2}$$

We sketch the proof. Set

$$P_n(D) = \prod_{k=1}^{n} \left(1 - \frac{D}{a_k}\right) e^{D/a_k}, \tag{5.3}$$

so that (5.1) is equivalent to

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} G_n(x - y)\phi(y)dy = \phi(x),$$

where

$$G_n(x) = P_n(D)G(x).$$

We have applied the operator (5.3) under the integral sign (2.1), a step easily justified with present hypotheses. By the classical inversion of (4.2), see [6, p. 241],

$$G(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{sx}}{E(s)} ds.$$

Again applying (5.3),

$$G_n(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{sx}P_n(s)}{E(s)} ds.$$

But the function

$$E_n(s) = \frac{E(s)}{P_n(s)}$$

itself belongs to E, so that Theorem 4.2 may be used to obtain the properties of $G_n(x)$. Set

$$\begin{aligned} I_n(x) &= \int_{-\infty}^{\infty} G_n(x - y)\phi(y)dy - \phi(x) \\ &= \int_{-\infty}^{\infty} G_n(y)[\phi(x - y) - \phi(x)]dy. \end{aligned}$$

For a fixed x and $\delta > 0$ write the integral $I_n(x)$ as the sum of two others $I_n'(x)$ and $I_n''(x)$ corresponding to the ranges of integration $|y| \leqq \delta$ and $|y| > \delta$, respectively. Then

$$| I_n'(x) | \leqq \max_{|y| \leqq \delta} | \phi(x - y) - \phi(x) |$$

by properties A and B of Theorem (4.2). Also

$$| I_n''(x) | \leqq 2 \sup_{-\infty < y < \infty} | \phi(y) | \int_{|t| > \delta} \frac{y^2}{\delta^2} G_n(y) dy$$
$$\leqq \frac{2}{\delta^2} \sup_{-\infty < y < \infty} | \phi(y) | \sum_{k=n+1}^{\infty} a_k^{-2},$$

by property D of Theorem 4.2. It is thus clear that $I_n'(x)$ can be made small by choice of δ, $I_n''(x)$ by choice of n, so that $I_n(x) \to 0$ when $n \to \infty$, as desired.

The choice $b = 0$ was inconsequential, amounting essentially to a choice of origin. A case in which $c \neq 0$ will be treated in §7.

6. **Application to the Laplace transform.** We now apply the foregoing theory to Example B of §2. But let us first recall the following real inversion [6, p. 288] of the Laplace integral (2.4),

$$\lim_{n \to \infty} \frac{(-1)^n}{n!} F^{(n)}\left(\frac{n}{y}\right)\left(\frac{n}{y}\right)^{n+1} = \Phi(y). \tag{6.1}$$

It is valid, whenever (2.4) converges, for almost all x and certainly at points where $\Phi(x)$ is continuous. For example, for the pair $\Phi(x) = e^{-x}$, $F(x) = (x+1)^{-1}$ it becomes $\lim_{n\to\infty} [1 + (y/n)]^{-n-1} = e^{-y}$. We shall show that (5.1) reduces to (6.1) in Example B. The function $E(x)$ in the present case is $1/\Gamma(1-x)$. For, the Laplace transform of the kernel is

$$\int_{-\infty}^{\infty} e^{-e^y} e^y e^{-xy} dy = \int_0^{\infty} e^{-y} y^{-x} dy = \Gamma(1 - x), \qquad -\infty < x < 1.$$

Since $1/\Gamma(1-x)$ is a function of class E with $c = 0$, Theorem 5 is applicable.[2] We need the familiar product expansion of the gamma function,

$$\frac{1}{\Gamma(1 - x)} = e^{-\gamma x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{k}\right) e^{x/k},$$
$$\gamma \sim \sum_{k=1}^{n} \frac{1}{k} - \log n, \qquad n \to \infty.$$

Set

[2] We really need a slight modification of Theorem 5 since $b = -\gamma$ in the present example.

$$P_n(x) = n^x \prod_{k=1}^{n} \left(1 - \frac{x}{k}\right),$$

so that (5.1) is equivalent to

$$\lim_{n \to \infty} P_n(D) e^x F(e^x) = \Phi(e^{-x}), \tag{6.2}$$

in the present case. But the left-hand side of (6.2) takes on an especially simple form arising from the equation

$$(1 - D) e^x F(e^x) = - e^{2x} F'(e^x).$$

By induction

$$\prod_{k=1}^{n} \left(1 - \frac{D}{k}\right) e^x F(e^x) = \frac{(-1)^n}{n!} e^{(n+1)x} F^{(n)}(e^x).$$

Of course n^D or $e^{(\log n)D}$ means translation of x through $\log n$ or multiplication of e^x by n. Hence

$$P_n(D) e^x F(e^x) = \frac{(-1)^n}{n!} n^{(n+1)x} F^{(n)}(n e^x),$$

and if $e^{-x} = y$, (6.2) becomes (6.1), as predicted.

7. **The Weierstrass transform.** If we denote the "source solution" of the heat equation by $k(x, t)$,

$$k(x, t) = (4\pi t)^{-1/2} e^{-x^2/4t}, \qquad -\infty < x < \infty,\ 0 < t < \infty,$$

then equation (2.5) becomes

$$f(x) = k(x, 1) * \phi(x). \tag{7.1}$$

Recalling that the inversion function predicted by the operational calculus is the reciprocal of the Laplace transform of the kernel and from (4.3) that

$$e^{s^2} = \int_{-\infty}^{\infty} e^{-sy} k(y, 1) dy, \tag{7.2}$$

we expect that

$$e^{-D^2} f(x) = \phi(x).$$

This operational equation was already observed by A. Eddington [2], who replaced e^{-D^2} by its Taylor development. However, this interpretation is usually ineffective because the resulting series diverges for most functions $f(x)$. We employ a different method.

Equation (4.3) is valid for all complex s, so that if we replace x by $-iD$ therein we obtain

$$e^{-tD^2} = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} e^{iyD}e^{-y^2/4t}dy.$$

Again using (3.1) with $a=ix$,

$$e^{-tD^2}f(x) = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} f(x+iy)e^{-y^2/4t}dy.$$

This integral is clearly a complex integral evaluated along a vertical line in the complex plane. If we set $x+iy=s$ it becomes

$$(7.3)\qquad e^{-tD^2}f(x) = -i(4\pi t)^{-1/2}\int_{x-i\infty}^{x+i\infty} f(s)e^{(s-x)^2/4t}ds.$$

For all functions $f(x)$ arising from equation (7.1) the integral (7.3) will be independent of the path of integration, and we accordingly make our definition more flexible by replacing the path of (7.3) by an arbitrary one.

DEFINITION 7. *The operator* e^{-D^2} *is defined as*

$$(7.4)\qquad \lim_{t\to 1-} e^{-tD^2}f(x) = \lim_{t\to 1-} -i(4\pi t)^{-1/2}\int_{c-i\infty}^{c+i\infty} f(s)e^{(s-x)^2/4t}ds,$$

where c is a suitable constant.

This definition should be compared and contrasted with (5.2). The continuous parameter t in (7.4) corresponds to the discrete one n in (5.2). The use of the parameter $t<1$ in (7.4) amounts to the use of Abel summability on an integral that would diverge for certain functions $f(x)$ when $t=1$.

We can now state our inversion result in terms of this operator. As in §5 we do so with simplifying but unnecessary restrictions on $\phi(x)$.

THEOREM 7. *If* $\phi(x)$ *is bounded and continuous on* $(-\infty, \infty)$ *and if* $f(x)$ *is its Weierstrass transform, given by* (7.1), *then*

$$(7.5)\qquad e^{-D^2}f(x) = \phi(x), \qquad -\infty < x < \infty.$$

We sketch the proof. Inverting equation (7.2), we have

$$k(x, 1) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{sx+s^2}ds.$$

If we compute the integral (7.4), $c=0$ when $f(x)$ is replaced by $k(x, 1)$, we have

$$e^{-tD^2}k(x, 1) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{sx+s^2}e^{-ts^2}ds,$$

and this is $k(x, 1-t)$ as we see by inverting (4.3). Hence after validating the application of e^{-tD^2} under the integral sign (Fubini's theorem) one obtains

$$(7.6) \qquad e^{-tD^2}f(x) = \int_{-\infty}^{\infty} k(x-y, 1-t)\phi(y)dy = k(x, 1-t) * \phi(x).$$

But this is the Weierstrass singular integral so familiar in the theory of heat conduction. It tends to $\phi(x)$ as $t \to 1-$, and (7.5) follows.

From the point of view of heat conduction $e^{tD^2}\phi(x)$ or $k(x, t)*\phi(x)$ is the temperature of an infinite rod t seconds after it was $\phi(x)$ and the Weierstrass transform of $\phi(x)$ is the temperature one second after it was $\phi(x)$. This makes (7.6) intuitive: $e^{-tD^2}f(x)$ is the temperature t seconds *before* it was $f(x)$ or $1-t$ seconds after it was $\phi(x)$.

8. **Table of inversion operators.** We conclude with a summary of the inversion operators mentioned in §2. Many others are treated in [4].

	Kernel $G(x)$	Inversion operator $E(D)$
A.	$g(x)$	$1-D$
B.	$e^{-e^x}e^x$	$1/\Gamma(1-D)$
C.	$(2\pi)^{-1}$ sech $(x/2)$	$\cos \pi D$
D.	$(4\pi)^{-1/2}e^{-x^2/4}$	e^{-D^2}
E.	$(2/\pi)K_0(e^x)e^x$	$\pi 2^D\Gamma^{-2}[(1-D)/2]$

In Example C, the fact that the Stieltjes transform is the iterated Laplace transform expressed in convolution form becomes

$$e^{x/2}F(e^x) = e^{-e^x}e^{x/2} * e^{-e^{-x}}e^{-x/2} * e^{x/2}\Phi(e^x).$$

The corresponding relation for the inversion functions $E(s)$ is the classical equation

$$\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{1}{2}+s\right) = \frac{\pi}{\cos \pi s}\cdot$$

Example E is the Meier transform.

REFERENCES

1. G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937.

2. A. Eddington, *On a formula for correcting statistics for the effects of a known probable error of observation*, Monthly Notices Royal Astronomical Society vol. 73 (1914) pp. 359–360.

3. I. I. Hirschman, Jr. and D. V. Widder, *A representation theory for a general class of convolution transforms*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 69–97.

4. ———, *The convolution transform*, A book to appear in the Princeton Mathematical Series.

5. I. J. Schoenberg, *On smoothing operations and their generating functions*, Bull. Amer. Math. Soc. vol. 59 (1953) pp. 199–230.

6. D. V. Widder, *The Laplace transform*, Princeton, 1941.

HARVARD UNIVERSITY