

REGULAR CONVERGENCE

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1. **Introduction.** The notion of convergence is certainly one of the most important ones in all mathematics. In analysis if a sequence of functions converges to a limit function, we ask the question what properties enjoyed by the members of the sequence are carried over to the limit function. With no restrictions on the type of convergence, of course very little can be said. We, therefore, impose conditions on the convergence which will allow some conclusions to be made. The same situation prevails in topology if we consider the concept of convergence of point sets. We say that the sequence of point sets (G_i) converges to the point set G , where all sets belong to a Hausdorff space, if the following is true. Every point with the property that each of its neighborhoods contains points from infinitely many G_i lies in G and each point of G has the property that each of its neighborhoods contains points from all but a finite number of the (G_i) . It is easily seen that G will always be closed regardless of whether the G_i are or not. This notion was first introduced by Zarankiewicz [1]. An equivalent definition in a compact metric space is as follows. If we call the spherical neighborhood of a point set X with radius ϵ the set of all points x whose distance from some point in X is less than ϵ , then (G_i) converges to G if G is closed and for every ϵ the spherical neighborhood about G with radius ϵ contains all but a finite number of the G_i and the spherical neighborhood with radius ϵ about all but a finite number of the G_i contains G . For example, the sets $G_i = \{(x, y) | x = 1/i, (0 \leq y \leq 1)\}$ converge to $G = \{(x, y) | x = 0, (0 \leq y \leq 1)\}$. The second definition is equivalent to saying that G is closed and the Hausdorff distances [2] from G_i to the G converge to 0. Very few properties are carried over to the limit set by convergence of this general type. The reason for this is that two sets can be close to each other without being at all similar. For example in the above mentioned example the sets G_i could be replaced by the points of the line forming G_i that are rational with denominator i , and the limit set would still be the same. The reverse situation is not true, however, i.e. if all the members of the sequence in a compact metric space are closed and connected, then the limit set will also be closed and con-

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nected. Thus under convergence several components may be put together, but not torn apart.

2. Regular convergence. By requiring more than just closeness in the convergence, a much more satisfactory situation can be achieved. In 1935 G. T. Whyburn gave such a definition [3] in which he made use of the definition of local connectivity. In his definition a sequence (A_i) of closed sets is said to *converge regularly* to a limit set A if for each $\epsilon > 0$ there exist positive numbers δ and N such that for $n > N$, any two points in A_n whose distance apart is $< \delta$ lie in a connected subset of A_n with diameter $< \epsilon$. If all the sets A_n in a sequence coincide with the limit set A , this is the definition of uniform local connectivity. Thus we require the members of the sequence to approximate each other more and more closely in the sense of local connectivity as the sequence progresses. With this type of convergence it is possible to prove the following results.

If (A_i) converges to A regularly, where all sets are contained in a compact metric space, then:

- (1) A is locally connected (whether any of the A_i are or not);
- (2) if in addition all sets A_i are locally connected continua, then every simple closed curve J and every simple arc ab in A (which is also a locally connected continuum) is the limiting set of a regularly convergent sequence of simple closed curves J_1, J_2, \dots or arcs a_1b_1, a_2b_2, \dots respectively, where $J_n, a_nb_n \subset A_n$;
- (3) if each A_i is an arc a_ib_i , then A is an arc ab (or a single point), and by a proper choice of notation $(a_i) \rightarrow a, (b_i) \rightarrow b$;
- (4) if each A_i is a simple closed curve, then A is a simple closed curve (or a single point);
- (5) if each A_i is a topological sphere, then A is a cactoid [4];
- (6) if each A_i is a 2-cell with boundary C_i such that $d(C_i) \rightarrow 0$ ($d(C_i) = \text{diameter of } C_i$), then A is a cactoid;
- (7) if each A_i is a 2-cell and their boundaries J_i converge to J , then J is a boundary curve—furthermore a necessary and sufficient condition for the convergence to be regular is that J be a simple closed curve;
- (8) if the A_i are 2-cells with boundaries J_i that converge to J , then A is a hemicactoid [4] whose base set is bounded by J .

3. r -regular convergence. The type of convergence described in §2 can be thought of as the zero-dimensional case of r -dimensional regular convergence. In order to give this definition it is necessary to make use of some kind of r -dimensional homology theory. In a

compact metric space, perhaps the most convenient theory is the one introduced by Vietoris [5].

3.1 DEFINITION. An ϵ -simplex of dimension r is a collection of $(r+1)$ -points whose diameter is less than ϵ .

3.2 DEFINITION. An ϵ -cycle (chain) of dimension r is a cycle (chain) of ϵ -simplices of dimension r .

3.3 DEFINITION. A cycle z^r is said to be ϵ -homologous to 0 ($z^r \sim_\epsilon 0$) if z^r is the boundary of an ϵ -chain C^{r+1} ($z^r = \partial C^{r+1}$).

3.4 DEFINITION. Two cycles z_1^r and z_2^r are ϵ -homologous to each other ($z_1^r \sim_\epsilon z_2^r$) if $z_1^r - z_2^r \sim_\epsilon 0$.

3.5 DEFINITION. An r -dimensional Vietoris cycle V^r is a collection of r -dimensional δ_i -cycles (z_i^r) called its coordinate cycles such that

- (1) $(\delta_i) \rightarrow 0$,
- (2) for each $\epsilon > 0$ there is an integer N such that for $m, n > N$, $z_m^r \sim_\epsilon z_n^r$.

3.6 DEFINITION. A Vietoris cycle V^r is homologous to 0 ($V^r \sim 0$) if for each $\epsilon > 0$ all but a finite number of its coordinate cycles are ϵ -homologous to 0.

3.7 DEFINITION. Two Vietoris cycles V_1^r and V_2^r are homologous to each other if for each $\epsilon > 0$ all but a finite number of the corresponding coordinate cycles are ϵ -homologous.

The coefficient group for the coordinate cycles is assumed to be the mod 2 group. However, many results hold with more general groups which will be noted from time to time. In no case will the coefficient group be more general than a commutative ring with a unit element.

We can now give the general definition of regular convergence of any dimension.

3.8 DEFINITION. The sequence of closed sets (A_i) is said to converge r -regularly to A if corresponding to each $\epsilon > 0$ there is a positive integer N and a $\delta > 0$ such that if $n > N$, any Vietoris cycle V^j , $j \leq r$, in a subset of A_n of diameter $< \delta$ is ~ 0 in a subset of A_n of diameter $< \epsilon$.

Here again if all the sets A_i were to coincide with A , then the above definition would become the usual one for uniform local- j -connectedness for all $j \leq r$ in terms of Vietoris cycles.

This definition of r -regular convergence was also introduced by G. T. Whyburn [3] except that he required the condition only on cycles of the one dimension r . However so little results without imposing the same conditions on the lower-dimensional cycles that it is easier to include all of them in the one definition.

It is easily seen that in the 0-dimensional case this definition is equivalent to the regular convergence of §2, since in a compact metric

space a V^0 can be interpreted as a pair of points and being ~ 0 is equivalent to the pair's lying in a connected subset. Thus we have a true generalization of the earlier definition.

In this type of convergence it is impossible to close up "holes" with r -dimensional boundaries that exist in the members of the sequence. For example suppose the set A_i is the set of polar coordinate points $\{(r, \theta) \mid r=1 \ (1/i \leq \theta \leq 2\pi - (1/i))\}$. The missing arc between $\theta = 1/i$ and $\theta = -1/i$ would be called a hole with a zero-dimensional boundary, i.e. the two end points of the arc. This hole is gradually closed up as i increases so that the limit set is the entire circle which contains no hole with 0-dimensional boundary. This could happen because the convergence was *not* 0-regular. This follows by considering the pair of end points $(1/i, 1)$ and $(2\pi - (1/i), 1)$ which may be thought of as 0-dimensional Vietoris cycles. The diameters of these cycles $\rightarrow 0$ as $i \rightarrow \infty$, but although each is ~ 0 in its set A_i , the diameter of the smallest set in which this cycle is ~ 0 (namely all of A_i) is always 2—violating the definition of 0-regular convergence. Next consider the sets A_i in the 3-dimensional space described in spherical coordinates as follows: $1/i \leq \rho \leq 1$ (i.e. all the points between and on the 2 concentric spheres $\rho = 1/i$ and $\rho = 1$). As i increases the hole inside bounded by the 2-dimensional boundary $\rho = 1/i$ is gradually shrunk so that the limit set is the solid sphere $0 \leq \rho \leq 1$. Here the 2-regular convergence is violated by the 2-cycles for the sphere $\rho = 1/i$ on A_i can be thought of as 2-dimensional Vietoris cycles. Now these cycles are contained in subsets of A_i whose diameters converge to 0, yet none is ~ 0 at all—contrary to the definition of 2-regular convergence. Notice, however, in this case that the convergence is 1-regular. Next consider the sets A_i in 3-dimensional space described in Cartesian coordinates as all points (x, y, z) such that $x^2 + y^2 + z^2 \leq 2$, and $x^2 + i^2 y^2 + i^2 z^2 \geq 1$. Again as i increases the sets A_i converge to a solid sphere $A: x^2 + y^2 + z^2 \leq 2$. However this time there are no nonbounding 2-cycles whose diameters converge to 0 as i increases. The convergence is still not 2-regular since the 1-cycles correspond to the circles $i^2 y^2 + i^2 z^2 = 1$ converge to 0 as i increases, but none is ~ 0 on a subset of the corresponding A_i with diameter < 1 . This time the convergence is 0-regular. Finally let $A_i =$ the set of points (x, y, z) such that $x^2 + y^2 + z^2 \leq 2$ and $x^2 + y^2 + z^2 i^2 \geq 1$, as i increases again (A_i) converges to the solid sphere $x^2 + y^2 + z^2 \leq 2$. This time the 2-regular convergence is *violated* because the 0-cycles correspond to the pairs of points $(0, 0, 1/i)$, $(0, 0, -1/i)$ have diameters that converge to 0 as i increases, but again do not bound on subsets with diameters < 1 . In this case the 2-regular convergence is *not* violated because of the

behavior of either the 1 or 2 cycles. Thus we see that a hole with a 2-dimensional boundary can be closed up if 2-regular convergence is violated by cycles of just one of the 3 dimensions 0, 1, or 2.

4. The Betti numbers and local connectivity. A more general and precise description of the behavior referred to in the previous section can be obtained in terms of the Betti numbers.

4.1 DEFINITION. In a compact metric space M the r -dimensional Betti number ($p^r(M)$) is the maximum number of linearly independent Vietoris cycles (i.e. a number k such that $a_1Z'_1 + \cdots + a_sZ'_s \sim 0$ for any $s > k$ Vietoris cycles Z'_1, \cdots, Z'_s , and elements a_1, \cdots, a_s from the coefficient group; but there exist k cycles Z'_1, \cdots, Z'_k such that $a_1Z'_1 + \cdots + a_kZ'_k \sim 0$ implies $a_1 = \cdots = a_k = 0$). If no such maximum number exists, we say that $p^r(M) = \infty$. Now a description of a "hole" with a 2-dimensional boundary can be given by saying that the 2-dimensional Betti number is one. To say that holes cannot be closed is included in the following theorem.

4.2 THEOREM. *If $(A_i) \rightarrow A$ r -regularly and $p^r(A_i) \leq n$ for all i , then $p^r(A) \leq n$.*

It is also true that holes cannot be formed under regular convergence according to the next theorem.

4.3 THEOREM. *If $(A_i) \rightarrow A$ r -regularly and $p^r(A) \leq n$, then $p^r(A_i) \leq n$ for all but a finite i .*

These theorems are due to H. A. Arnold [6].

In the case of 0-regular convergence, we saw that the limit set is always locally connected. The following theorem states the general result due to G. T. Whyburn [7].

4.4 THEOREM. *If $(A_i) \rightarrow A$ r -regularly, and each A_i is lc^r (locally- j -connected for all $j \leq r$), then A is lc^r .*

In case the coefficient group is an arbitrary ring with a unit, E. G. Begle has shown [8] that the above theorem is not always true; but if each A_i is also required to be lc^n , then the theorem is true.

5. Manifolds. We have already seen that under 0-regular convergence a sequence of the simplest manifolds, i.e. simple closed curves or arcs, has for its limit a manifold of the same kind (if non-degenerate). It is natural to consider the behavior of higher-dimensional manifolds under the higher-dimensional regular convergence. However it is well known that no topological characterizations exist for the manifolds of dimension higher than 2; therefore, it is necessary

to consider generalized manifolds instead of the classical ones. A generalized manifold is a space which has many of the same homology characteristics as a classical manifold of the same dimension and reduces to the classical case for dimensions ≤ 2 . The definition we shall use is due to R. L. Wilder [9].

5.1 DEFINITION. The compact n -dimensional (Menger-Urysohn dimension—[10]) metric space M is called a *closed (orientable) n -dimensional manifold* if:

- (1) $\phi^n(M) = 1$, but $\phi^n(F) = 0$ for every proper closed subset F of M .
- (2) If $p \in M$, there exists an $\epsilon > 0$ such that if V^i ($1 \leq i \leq n-1$) is an i -cycle of the sphere $S(p, \epsilon)$, then $V^i \sim 0$ in M .
- (3) If $p \in M$, and $\epsilon > 0$ are arbitrary, then there exist positive numbers δ and η , $\epsilon > \delta > \eta$, such that if V^i ($0 \leq i \leq n-2$) is a cycle of the boundary of the sphere $S(p, \delta)$, then $V^i \sim 0$ in $S(p, \epsilon) - S(p, \eta)$; if V^{n-1} is a cycle of this boundary, then $V^{n-1} \sim 0$ in $M - S(p, \eta)$.

By using cohomology theory E. G. Begle has given a definition [11] that is essentially equivalent. It is easily seen that the simple closed curve and 2-sphere satisfy Definition 5.1 for dimensions 1 and 2, respectively. Conversely, Wilder has shown that a 1- or 2-dimensional generalized manifold must be a classical one.

Begle has proved the following theorem [8].

5.2 THEOREM. *If $M_i \rightarrow M$ ($r-1$)-regularly, where each M_i is a closed r -dimensional orientable generalized manifold and M is r -dimensional, then M is a closed r -dimensional orientable generalized manifold.*

This theorem was proved in the case where the coefficient group for cycles can be any commutative ring with a unit.

In the 2-dimensional cases the following theorems can be stated. The first was announced by H. E. Vaughan [12] and also proved by Begle [8].

5.3 THEOREM. *Let (M_i) be a sequence of closed, orientable 2-dimensional manifolds and let (M_i) converge 1-regularly to a nondegenerate set M . Then M is a closed, orientable 2-dimensional manifold, and for all sufficiently large i , M and M_i are homeomorphic.*

The author has also defined a generalized manifold with boundary [29] and the following theorem is proved [30].

5.4 THEOREM. *If S is an orientable n -dimensional generalized closed manifold and $K \subset S$ is an $(n-1)$ -dimensional generalized closed manifold such that $S - K = A \cup B$ separate, and K is the common boundary of A and B , then \bar{A} and \bar{B} are n -dimensional generalized closed manifolds with boundary K .*

For our purposes the result of this theorem may be used as the definition of a manifold with boundary, and the author has proved the following result [31].

5.5 THEOREM. *If (M_i) is a sequence of orientable generalized n -manifolds with boundaries (K_i) , such that $M_i \rightarrow M$ $(n-1)$ -regularly and $K_i \rightarrow K$ $(n-2)$ -regularly, then M is an orientable generalized n -manifold with boundary K .*

In the 2-dimensional case it is proved as in 5.3 that all but a finite number of the M_i are homeomorphic with M . Whyburn first proved the case where each M_i is a 2-cell, but used the following theorem to do it.

5.6 THEOREM. *If a sequence of 2-cells converge 1-regularly to a limit set, then the limit set is a base set.*

6. An alternate definition of 0-regular convergence and its generalization. Whyburn has given the following interesting characterization of 0-regular convergence [3].

6.1 THEOREM. *Let the sequence of closed sets (M_n) converge to the limiting set M . Then in order that (M_n) converge 0-regularly to M it is necessary and sufficient that for each sequence of decompositions $M_{n_i} = A_{n_i} + B_{n_i}$ into closed sets such that $(A_{n_i}) \rightarrow A$, $(B_{n_i}) \rightarrow B$, and $(A_{n_i} \cdot B_{n_i}) = (X_i) \rightarrow X$, we have $A \cdot B = X$.*

No similar characterization of the higher-dimensional cases is known, but the following similar theorems were announced by H. A. Arnold [6].

6.2 THEOREM. *If $(M_n) \rightarrow M$ r -regularly, then for every sequence of decompositions $M_{n_i} = A_{n_i} + B_{n_i}$ into closed sets, such that the sequence $(X_i) = (A_{n_i} \cdot B_{n_i}) \rightarrow X$ and the sequences (B_{n_i}) , $(A_{n_i}) \rightarrow (r-1)$ -regularly to B and A , respectively, then $(X_i) \rightarrow X$ $(r-1)$ -regularly.*

6.3 THEOREM. *If $(M_n) \rightarrow M$ r -regularly, and $(A_n \cdot B_n) \rightarrow A \cdot B$ r -regularly, then $(A_n) \rightarrow A$ and $(B_n) \rightarrow B$, both r -regularly.*

7. Regular convergence in terms of Čech cycles. The use of Vietoris cycles ties the work to a metric space. If one wishes to break away from this restriction, the Čech theory of cycles in terms of coverings of the space by open sets can be used. The definition of local- r -connectivity, r -dimensional Betti number, generalized manifold, etc., can all be rephrased in terms of this theory with only the assumption of bicomactness (every open covering can be reduced to a finite sub-covering). These definitions are all given in R. L. Wilder's Col-

loquium publication [13]. Here the coefficient group is assumed to be an arbitrary field.

The author has proved [14] that the theorems regarding the Betti numbers, local connectivity, and generalized manifolds, 4.2, 4.3, 4.4, and 5.2, still hold in this case.

In proving the above results the following theorem concerning the existence of a normal refinement was proved.

7.1 THEOREM. *If $(A_i) \rightarrow A$ r -regularly, then corresponding to any covering of the space there is a covering which is a normal refinement of it relative to r -dimensional Čech cycles of A_i for any i .*

Wilder defines and proves the existence of such a covering for one set [13], but to the author's knowledge this is the only known theorem on the existence of such a refinement for an infinite collection of sets.

8. Regular convergence spaces. The collection of all closed lcr subsets of a compact metric space M can be made into a hyperspace K^r by defining the notion of convergence by means of regular convergence. Thus we shall say that the sequence of points (a_i) of the hyperspace converges to the limit point a if their corresponding subsets (A_i) converge r -regularly to the set A corresponding to a . It is clear from Theorem 4.4 that the space K^r is closed relative to the similarly constructed hyperspace of all closed subsets of M . Also by 4.2 and 4.3 the set K_n^r consisting of all points of K^r corresponding to sets whose r -dimensional Betti number is n is closed. It is shown by the author [15] that

(8.1) K^r is metrizable and separable.

Begle also obtained this result independently [8] and showed that

(8.2) K^r is topologically complete.

Many point set properties can be related to special subsets of K^r as stated in the following theorems:

8.3 *If $r > 0$, then a necessary and sufficient condition that M contain no simple closed curve is that K_1 (the subspace of K^r corresponding to locally connected continua) be connected. In case M is locally connected this is a necessary and sufficient condition for a dendrite [4].*

8.4 *A necessary and sufficient condition that every convergent sequence of arcs in M converge 0-regularly is that K_1 be compact.*

8.5 *If M is compact, then a necessary and sufficient condition that every convergent sequence of arcs converge 0-regularly is that corresponding to every $\epsilon > 0$ there exists a $\delta > 0$ such that if p and q are 2 points of M whose distance apart is $< \delta$, then every arc joining p and q has diameter $< \epsilon$.*

8.6 If M is a Euclidean cube of dimension $\geq r+2$ and of diameter 1, then the subset N of K^r consisting of those subsets of M which are lc^{r+1} is of the 1st Baire category in K^r .

9. Relation to monotone transformations.

9.1 DEFINITION. If A is compact metric and the transformation $T(A)=B$ is continuous, then T is said to be r -monotone if for each $b \in B$, $p^i(T^{-1}(b))=0$, $i \leq r$. In the 0-dimensional case this says that the inverse of every point is connected and reduces to the usual definition of a monotone transformation. It has been shown [4] that the monotone image of

- (1) an arc is an arc,
- (2) a simple closed curve is a simple closed curve (or a point),
- (3) a topological sphere is a cactoid,
- (4) a 2-cell with boundary J is a hemicactoid with boundary curve $f(J)$.

Also Wilder has shown [22] that

- (5) the r -monotone image of a closed $(r+1)$ -dimensional orientable generalized manifold is also a closed $(r+1)$ -dimensional orientable generalized manifold.

Notice that if $(A_i) \rightarrow A$ 0-regularly and if each A_i is a set of type 1, 2, 3, or 4, we have already stated that A will also be a set of the same type. Also if the convergence is r -regular and each A_i is of type 5, then A is also of that type. This suggests that there may be a relationship between monotone transformations and regular convergence. The following theorem due to Whyburn [7] demonstrates this relationship.

9.2 THEOREM. Let the sequence of r -monotone transformations $T_i(A) = B_i$ converge uniformly to the limit transformation $T(A) = B$. In order that the sequence $(B_i) \rightarrow B$ r -regularly, it is necessary and sufficient that T be s -monotone, $s \leq r$, and B be an lc^r .

10. Regular transformations. Suppose A and B are compact metric spaces and the continuous mapping T carries A onto B . T is called interior [16] if it carries open sets into open sets. Eilenberg showed [17] that T is interior in this case if and only if for each sequence $(b_i) \rightarrow b$ in B , $(T^{-1}(b_i)) \rightarrow T^{-1}(b)$. A natural generalization is to require the convergence to be r -regular in which case the transformation is called r -regular. The 0-dimensional case was first studied by Wallace [18] and the r -dimensional case by Puckett [19] and the author [20]. Puckett showed that if T is $(n-1)$ -regular, then for any two points b and b' of B the Betti groups of dimensions $\leq n$ of $T^{-1}(b)$ and $T^{-1}(b')$ are isomorphic. The author has shown (a) that under an $(n-1)$ -

regular transformation every small Vietoris cycle in B of dimension $\leq n$ is the image of a small cycle in A . Also (b) that if the transform of a cycle of dimension $\leq (n-1)$ in A is ~ 0 in a small subset of B , then the original cycle is ~ 0 in a small subset of A . From proposition (a) the following results can be proved. In each case assume $T(A) = B$ is $(n-1)$ -regular.

10.1 THEOREM. *If the sequence of closed sets $(Y_i) \rightarrow Y$ in B , then $(T^{-1}(Y_i)) \rightarrow T^{-1}(Y)$ and if the latter convergence is n -regular so is the former.*

10.2 THEOREM. *If T is factored, $T = T_2 T_1$, so that $T_1(A) = C$ is $(n-2)$ -regular, then $T_2(C) = B$ is $(n-1)$ -regular.*

10.3 THEOREM. *If A is lc^n , then B is lc^n .*

10.4 THEOREM. *If A is a continuum, then T can be factored, $T = T_2 T_1$, so that $T_1(A) = C$ is monotone and $(n-1)$ -regular and $T_2(C) = B$ is of constant multiplicity and locally topological.*

From proposition (b) the following results are obtained:

10.5 THEOREM. *In order that $T(A) = B$ be n -regular it is necessary and sufficient that for any sequence of closed sets $(Y_i) \rightarrow Y$ n -regularly in B , we have $(T^{-1}(Y_i)) \rightarrow T^{-1}(Y)$ n -regularly.*

10.6 THEOREM. *If $T(A) = B$ is n -regular and $Y \subset B$ is lc^n , then $T^{-1}(Y)$ is lc^n .*

10.7 THEOREM. *If $T_1(A) = C$ and $T_2(C) = B$ are n -regular, then so also is $T = T_2 T_1$.*

Puckett proved the following interesting theorems [19; 21] in the 0-regular case, where in each case we assume $T(A) = B$ is 0-regular and monotone (i.e. the first factor guaranteed by 10.4 above).

10.8 THEOREM. *If A is a continuum, then its 1-dimensional Betti group is the direct sum of 2 groups U and V , where U is isomorphic to the 1-dimensional Betti group of B and V is isomorphic to the Betti group of each set $T^{-1}(b)$ (b is any point of B).*

10.9 THEOREM. *If A is a 2-dimensional pseudo-manifold (i.e. a classical 2-dimensional manifold with or without boundary among q points of which identifications have been performed to produce r local separating points of A), then the following situations can occur and only these:*

- (1) T is topological or B is a single point;

$$(2) A = \left. \begin{array}{l} \textit{sphere} \\ \textit{2-cell} \\ \textit{circular ring} \end{array} \right\}, B = \textit{arc};$$

$$(3) A = \left. \begin{array}{l} \textit{torus} \\ \textit{Klein bottle} \\ \textit{circular ring} \\ \textit{Möbius band} \\ \textit{pinched sphere} \\ \textit{2-cell with 2 boundary} \\ \textit{points identified} \end{array} \right\}, B = \textit{simple closed curve}.$$

Furthermore in the above theorem the topological behavior of each transformation can be precisely described. In (2) of the theorem if A is the sphere $x^2 + y^2 + z^2 = 1$, then T is essentially the mapping that carries the circle cut by $z = k$ ($-1 \leq k \leq 1$) into the point $(0, 0, k)$ on the z -axis. If A is the 2-cell $x^2 + y^2 \leq 1$, then T is essentially the mapping that carries the segment cut by $y = k$ ($-1 \leq k \leq 1$) into the point $(0, k)$ on the y -axis. If A is the circular ring $1 \leq \rho \leq 2$ (polar coordinates), then T is essentially the mapping that carries the circle $\rho = k$, $1 \leq k \leq 2$, into the point $\rho = k$, $\theta = 0$. In (3) if A is the torus obtained from the cylinder $x^2 + y^2 = 1$, $-1 \leq z \leq 1$, by identifying the circles corresponding to $z = -1$ and $z = 1$, then T is essentially the mapping that carries the circle corresponding to $z = k$ into the point $(0, 0, k)$ of the simple closed curve obtained from the line segment $x = 0, y = 0, -1 \leq z \leq 1$, by identifying $(0, 0, 1)$ and $(0, 0, -1)$. If A is the Klein bottle obtained from the same cylinder by identifying the ends with opposite orientations, then the same transformation as before can be used. If A is the circular ring, $1 \leq \rho \leq 2$ (polar coordinates), then T is essentially the mapping that carries the segment $\theta = k$, $1 \leq \rho \leq 2$, into the point $\theta = k$, $\rho = 1$. If A is the Möbius band obtained by cutting the above ring along $\theta = 0$ and applying a twist before sewing it back together, then the same T as before can be used. If A is the pinched sphere obtained by rotating the circle $(x-1)^2 + z^2 = 1$ about the z -axis, then T is essentially the mapping that carries the circle generated by the point $(x, \pm(1-(x-1)^2)^{1/2})$ into the point $(x, \pm(1-(x-1)^2)^{1/2})$. Finally if A is the 2-cell with 2 boundary points identified obtained by rotating the circle $(x-1)^2 + z^2 = 1$ one-half a revolution about the z -axis, then T is the same as in the preceding case with half circles going into points.

11. Other types of regular convergence. We pointed out in §2 that regular convergence was derived from the definition of local connectedness by stringing the property out over the sets of a convergent

sequence. This same technique can be used on other local properties such as (a) r -dimensional semi-local connectedness [23], (b) complete r -avoidability [24], and (c) homotopy local connectedness for dimensions $\leq r$ (LC^r) [25]. The convergence obtained by using property (a) is called r -dimensional coregular convergence [26], that obtained from property (b) is called r -c.a. regular convergence [26], and that obtained from (c) homotopy- r -regular convergence [27]. It is shown [26] that if the convergence of a sequence of sets is $(r-1)$ -regular and r -coregular then the limit set contains no r -cut points [24]. If the convergence is n -regular, i -c.a. regular for $i \leq n-2$, and $(n-1)$ -coregular, where the members of the sequence are each n -dimensional closed Cantorian manifolds [28], then the limit set is a closed (orientable) n -dimensional generalized manifold (Definition 5.1). It is shown [26] that in the 0-dimensional case, 0-c.a. convergence implies both 0-regular and 0-coregular convergence for continua. Also a sequence of 2-dimensional compact classical manifolds converge 0-c.a. regularly to another such manifold (or a point) and if each member of the sequence has n disjoint simple closed curves as boundary, then the limit set has $m \leq n$ simple closed curves as its boundary—if each member is a sphere with n handles, then the limit set is a sphere with $m \leq n$ handles. If the sequence converges homotopy- p -regularly where all members of the sequence and the limit set are LC^p (dimension $\leq p$), then it is shown [27] that infinitely many members of the sequence have the same homotopy type as the limit set. Notice that it was not asserted that LC^p for the members of the sequence implies the same property for the limit sets, however a similar result can be shown [27].

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