the Treatise of Watson (1922); they are due to Watson, van der Corput, Langer, and others. The same is true for the section on the zeros. The second part contains a huge system of formulas.

Chapter VIII (T), Functions of the parabolic cylinder and of the paraboloid of revolution.

Chapter IX (M, T), The incomplete gamma functions and related functions.

Chapter X (T), Orthogonal polynomials. Various parts of this chapter go considerably beyond the material dealt with in the reviewer's book on the same topic: for instance, the work of Tricomi on the asymptotic behavior and the zeros of Laguerre polynomials and the properties of the important polynomials of Pollaczek.

Chapter XI (M), Spherical and hyperspherical harmonic polynomials, a very elegant and important chapter much of which was taken from unpublished notes of a course given by G. Herglotz.

Chapter XII, Orthogonal polynomials in several variables.

Chapter XIII (T), Elliptic functions and integrals.

Both volumes have a subject index and index of notations which will greatly increase the usefulness of the work.

The mathematical public will be indebted to the collaborators and to the editor of this project for their accomplishment.

G. Szegö

The algebraic theory of spinors. By C. Chevalley. New York, Columbia University Press, 1954. 8+131 pp.

Most of the results of the theory of spinors are due to its founder E. Cartan; and, until this year, the only place where they could be found in book form was E. Cartan's own *Leçons sur la théorie des spineurs*, published in 1938. Strangely enough, the deep and unerring geometric insight which guided Cartan's researches, and places him among the greatest mathematicians of all time, is too often smothered in his books under complicated and seemingly gratuitous computations: witness, for instance, his fantastic definition of spinors (at the beginning of the second volume of the work quoted above) by means of the coefficients of a system of (non-independent) linear equations defining a maximal isotropic subspace! The reason for this is most probably to be found in the fact that E. Cartan's generation did not have at its disposal the geometric language which modern linear algebra has given us, and which now makes it possible to express in a clear and concise way concepts and results which otherwise would remain hopelessly buried under forbidding swarms of matrices.

The remarkably skillful way in which this language is used is certainly the most conspicuous feature of Chevalley's book. It goes without saying that, as usual in modern algebra, the basic field K of the theory is arbitrary (whereas E. Cartan considered only the real and complex number fields of classical analysis); the specialist will, however, be amazed to see that the author has succeeded in pushing this generality so far that in the greater part of the book, no special treatment is necessary for fields of characteristic 2; and indeed, so great is the author's virtuosity that the non-specialist will need a very thorough reading of the book to realize that this case actually exhibits special features at all.

The first chapter begins with the fundamental properties of the orthogonal groups: as mentioned above, characteristic 2 is included from the start, and all the necessary results are proved in 14 pages, including the best proof of Witt's theorem known to the reviewer, and a new proof of the generation of the orthogonal groups by symmetries (the author extends that name to the orthogonal transvections in the case of characteristic 2; the justification for this is of course that both can be given the same definition and handled in exactly the same way). The second half of the chapter is devoted to the study of the representations of the orthogonal group on the p-vectors, their decompositions into simple components and the classification of these with regard to equivalence: this is done not only for the orthogonal group, but also for the subgroup of rotations and the group of commutators; for characteristic 2, only the case $p=1$ is considered (the representation being no longer completely reducible for $p>1$).

The first part of chapter II gives a complete study of the Clifford algebra of a quadratic form, and can be considered as the first such study in the literature, for all other books on spinors or quadratic forms are in such a hurry to reach their main theme that they are content with giving the Clifford algebra the most cursory treatment, brought down to the minimum number of properties they really need. Chevalley's presentation of the theory is entirely original; the main novelty consists in exhibiting a fundamental connection between the Clifford algebra C and the exterior algebra E of the underlying vector space M (it has long been noticed that the two algebras exhibit very similar features, but this had remained very vague until now). The quadratic form $Q(x)$ being written as $B_0(x, x)$, where B_0 is a symmetric bilinear form, Chevalley shows that C, as a vector space, can be identified with E, and that the multiplication in C can be obtained from the multiplication in E and the form B_0 in the following explicit way: it is sufficient to define multiplication on the left by an element $x \in M$ (since they generate C); the operator $L_x' : s \rightarrow xs$ can then be written $L_x' = L_x + \delta_x$, where L_x is the operator of left multiplication by x in E, and δ_x the (unique) antiderivation of E such that $\delta_x \cdot y = B_0(x, y)$ for every $y \in M$. This is the cornerstone on which all subsequent developments are based. First the simplicity of C when M has even dimension $n = 2r$ and Q is nondegenerate (and nondefective when the characteristic is 2) is proved by reducing the question to the case in which Q has maximal index r, and then exhibiting a faithful representation $w \rightarrow \rho(w)$ of C onto the ring of vector-space endomorphisms of an isotropic subspace N of M, of maximal dimension r (a method which will acquire fundamental importance in chapters III and IV). From this follow easily the structure of the subalgebra C^+ consisting of elements of even order in C, relations between C and the Clifford algebras of the restrictions of Q to two supplementary orthogonal subspaces of M, the structure of C and C^+ when n is odd, and finally the determination of the radical of C when Q is degenerate or defective.

The classical connection between C and the orthogonal group $O_n(K, Q)$ (Q is henceforth taken as nondegenerate and nondefective) is then developed: to avoid trouble with such ill-defined concepts as "many-valued representations," Chevalley starts with the group Γ of invertible elements $s \in C$ such that $sMs^{-1} = M$, and shows that $x \rightarrow sxs^{-1}$ is a transformation $\chi(s)$ of the orthogonal group O_n, χ being a mapping of Γ onto O_n (with a single exception, when K has 2 elements, $n = 4$ and the index of Q is 2). To the subgroup Γ^+ of even elements of Γ corresponds the group of rotations O_n^+; on the other hand, if $t \rightarrow t^J$ is the natural anti-automorphism of C (associating to a product $x_1 x_2 \cdots x_p$ of elements of M the product $x_p x_{p-1} \cdots x_1$ taken in the reverse order) the elements $s \in \Gamma^+$ have a "norm" $\lambda(s) = ss^J$ in K, and those having norm 1 form a normal subgroup Γ_0^+ of Γ^+, which is mapped onto a normal subgroup O_n' of O_n^+, containing the commutator subgroup Ω_n of O_n. Following Eichler, it is proved that for forms Q of index $\nu > 0$, $O_n' = \Omega_n$ and O_n^+/Ω_n is isomorphic to the multiplicative group of elements of K, modulo the squares in K.

Spinors are next introduced, as forming a space in which (for n even) acts a simple representation ρ of C; the restriction ρ^+ of ρ to C^+ is either simple or splits into two simple nonequivalent representations, the *half-spin* representations. Similar definitions are given in the odd-dimensional case, and the following sections study the restriction of the spin representation of C to the Clifford algebra of the restriction of Q to a non-isotropic subspace, and its extension when the base field K is extended. The chapter ends with a study of the classical case of quadratic forms over the real field (with special emphasis on the relationship between the Clifford algebra and the Lie algebra of the orthogonal group), and with a very elegant proof of Hurwitz's theorem on quadratic forms "permitting composition," using the simplicity of the Clifford algebra.

Chapters III and IV are restricted to the special case of quadratic forms of maximal index $[n/2]$; no attempt is made to extend the results obtained in that case (in particular, the principle of triality) to more general ones, and this is probably the only part of E. Cartan's theory which is not covered by the book. Chapter III begins with the exposition of the theory of *pure spinors*, one of the most beautiful discoveries of E. Cartan, which unfortunately also constitutes one of the most obscure parts of his book. Here everything is neatly cleared up by Chevalley; the dimension $n=2r$ being even, the space M is decomposed into a direct sum of two totally isotropic subspaces N and P, and the space S of spinors is identified with the subalgebra C^N of C generated by N (and isomorphic to the exterior algebra of N); if f is an r-vector representing P, $Cf=C^Nf$ is a minimal ideal, and the spin representation ρ is defined by $vuf=(\rho(v)\cdot u)f$ for $u\in C^N=S$, $v\in C$. Now for every maximal isotropic subspace Z, let f_Z be the product in C of the elements of a base of Z; $f_Z C$ is a minimal right ideal of C, and its intersection with Cf is a 1-dimensional vector subspace; any element of that space can be written $u_Z f$ where u_Z is a spinor well determined up to a scalar factor, and these spinors are the *pure spinors* associated to Z. Such a spinor entirely determines Z, as the set of vectors x such that $\rho(x)\cdot u_Z=0$, and conversely this condition is characteristic for the pure spinors associated to Z. Pure spinors play for maximal isotropic subspaces a part similar to the one which decomposable p-vectors play for p-dimensional vector spaces in exterior algebra. Their study is developed in great detail: they are always half-spinors, and the two families of pure half-spinors correspond to the two intransitivity classes of maximal isotropic spaces under the group of rotations; a sum $u+u'$ of two pure spinors is pure if and only if the intersection of their corresponding subspaces has dimension $r-2$. An interesting feature, which is an original contribution of the author, is an expression of the elements $s\in\Gamma$ such that $\chi(s)$ leaves all elements of N invariant; s can be written uniquely in the form $\exp(u)$, where $u=\sum_{i<j}a_{ij}x_ix_j$ is a 2-vector in N (the x_i's being a base of N), and $\exp(u)=\prod_{i<j}(1+a_{ij}x_ix_j)$ by definition. Using this, the author can show that a pure spinor corresponding to a maximal

isotropic subspace Z can be written $\exp(u)x_1x_2 \cdots x_h$, where the x_i's form a base for $Z \cap N$.

Next there is introduced, after Cartan, the bilinear invariant $\beta(u, v)$ on $S \times S$, as being the scalar such that $(uf)^J vf = \beta(u, v)f$; its invariance is expressed by the equation

$$\beta(\rho(s) \cdot u, \rho(s) \cdot v) = \lambda(s)\beta(u, v)$$

for any $s \in \Gamma$. It is shown that β is a nondegenerate bilinear form, which is either symmetric or antisymmetric according to the parity of $r(r-1)/2$; and $\beta(u, v) = 0$ for pure spinors u, v is the condition for their corresponding subspaces to have an intersection not reduced to 0.

The following sections are devoted to the study of the tensor product of the spin representation ρ by itself. First the tensor product $S \otimes S$ of the space S by itself can be identified to C, by the linear mapping $\phi(u \otimes v) = ufv^J$, and this immediately shows that the tensor product $\rho \otimes \rho$ can be identified with the representation which, to each $s \in \Gamma$, assigns the endomorphism $w \rightarrow \lambda(s)sws^{-1}$ of the vector space C. The most complete results are obtained in the case of characteristic $\neq 2$; then one can choose the form B_0 in an intrinsic way, as $B(x, y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$, and it can be shown that with this particular identification of C to the exterior algebra E, any automorphism of C is also an automorphism of E. In particular, an inner automorphism of C leaves invariant the subspaces of p-vectors when it leaves M invariant, i.e. when it is determined by an element $s \in \Gamma$. This gives immediately the decomposition of $\rho \otimes \rho$ in a direct sum of representations in the spaces of multivectors, studied in chapter I. To this decomposition corresponds a decomposition of ufv^J, for u and v in S, into the sum $\sum_{h=0}^{n} \beta_h(u, v)$, where β_h is a bilinear mapping of $S \times S$ into the space E_h of h-vectors, which is covariant under the representation $\rho \otimes \rho$. The study of these mappings enables one to describe completely the decomposition of the representation $\rho \otimes \rho$ when restricted to the group Γ^+; they also yield a characterization of pure spinors, and a criterion giving the dimension of the intersection $Z \cap Z'$ of two maximal isotropic subspaces, in terms of the corresponding pure spinors.

The remaining sections of chapter III are taken up by the relations between the half-spin representations of Γ^+ and their restrictions to the subgroup leaving invariant the elements of a nonisotropic plane, the determination of the kernels of the half-spin representations, the extension of the theory to odd dimensional spaces (by imbedding the space as a hyperplane in an even-dimensional space), and finally an application of spinor theory to get the classical description of the orthogonal group in 6 variables when the index is 3 (as isomorphic to a linear group in 4 variables): this does not seem to the reviewer to bring any information which may not be obtained in a quicker and more natural way by the classical method.

Chapter IV develops the famous "principle of triality." The dimension being $2r=8$, and the index equal to 4, the spaces S_p, S_i of half-spinors have the same dimension 8 as M. Following Cartan, the direct sum $A=M+S_p+S_i$ is considered, and on it are defined: 1° a symmetric bilinear form $\Lambda(x+v, x'+v')=B(x, x')+\beta(v, v')$ for x, x' in M, v, v' in S; 2° a trilinear symmetric form $\phi(\xi, \eta, \zeta)$ such that $\phi(x, u, u')=\beta(\rho(x)\cdot u, u')$ for $x\in M$, $u\in S_p$, $u'\in S_i$. From these one defines a (non-associative, but commutative) multiplication $\xi \circ \eta$ in A, by the condition $\phi(\xi, \eta, \zeta)=\Lambda(\xi \circ \eta, \zeta)$. All these definitions are invariant under the group Γ_0 (subgroup of the $s\in\Gamma$ such that $\lambda(s) = ss^J=1$) and conversely any automorphism of A which leaves invariant each of the subspaces M, S is produced by an element of Γ_0. But in addition, there is an automorphism j of A, of order 3, which permutes M, S_p, and S_i cyclically, and the existence of such an automorphism constitutes the principle of triality; it can be shown that for $x\in M$, $j(x)$ is of the form $u_1 \circ x\in S_i$ and $j^{-1}(x)=u_1' \circ x\in S_p$, where u_1 is a fixed semi-spinor in S_p and u_1' a fixed semi-spinor in S_i. Beautiful geometric interpretations of the multiplication $\xi \circ \eta$ and of the automorphism j can be given when they act on pure spinors. Finally, the mapping $(x, y)\rightarrow x * y=(x \circ u_1') \circ (y \circ u_1)$ defines a nonassociative multiplication in M itself, which is shown to be that of the Cayley-Dickson algebra of *octonions*; on the other hand, j defines in a natural way an automorphism $\bar{j}$ of the commutator subgroup Ω_8, and the subgroup of Ω_8 consisting of the invariant elements under that automorphism constitutes the group of automorphisms of the algebra of octonions. At this point the stage is set for the geometric study of the exceptional Lie groups, in which the author has recently made such remarkable progress (in work unfortunately still partly unpublished); and it is to be hoped that in the near future, taking up the task where he breaks it off here, he will lead us into this fascinating new geometry and thus add to the thanks he has deserved from all mathematicians for the splendid job he has done in this volume.

The proofreading has not been too careful, and a list of corrections would be welcome, as also an index of notations.

J. Dieudonné