THE APRIL MEETING IN NEW YORK

The five hundred first meeting of the American Mathematical Society was held at Columbia University in New York City on Friday and Saturday, April 23–24, 1954. About 375 persons registered, including 335 members of the Society.

In this meeting the Society associated itself with the theme of Columbia's Bicentennial Celebration—Man's Right to Knowledge and the Free Use Thereof. In celebration of this anniversary a special session was held Friday evening in the McMillin Theatre at which Professor John von Neumann delivered an address entitled *The mathematical method* upon joint invitation of the University and the Society. President Whyburn, who presided, presented a plaque to Columbia felicitating the University on its two centuries of devotion to the furtherance of wisdom and the enlightenment of mankind.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor Harry Pollard of Cornell University addressed the Society Friday afternoon on *Fundamental sets of functions* and Professor F. I. Mautner of the Johns Hopkins University addressed the Society Saturday afternoon on *Fourier analysis and the theory of groups*, at sessions presided over by Professor J. F. Randolph and Professor J. L. Walsh respectively.

Sessions for contributed papers were held Friday afternoon and Saturday morning and afternoon, Professors C. E. Rickart, Eugene Lukacs, Y. W. Chen, R. D. Schafer, I. M. Sheffer, and T. W. Moore presiding.

The abstracts of contributed papers presented at the meeting follow. Those with "*t*" after the abstract number were presented by title. Where a paper with joint authorship was presented in person, (p) follows the name of the author presenting it. Dr. Moser and Mr. Shenitzer were introduced by Professor Wilhelm Magnus, Dr. Gaier by Professor J. L. Walsh, Dr. Peyerimhoff and Dr. Jurkat by Professor C. N. Moore, Mr. Maximon by Professor G. W. Morgan, Dr. Burrow by Professor Edward Rosenthal, Mr. McAuley by Professor F. B. Jones, Mr. Linis by Dr. G. H. M. Thomas, Dr. Hintikka by Dr. Hartley Rogers, and Miss Heller by Dr. Isidor Heller.

ALGEBRA AND THEORY OF NUMBERS

376t. H. W. Becker: *Automedian tetrahedrons.* Preliminary report.

Let $p^2+q^2=2z^2$, $p^2+r^2=2y^2$, $q^2+r^2=2x^2$. Euler's (2), p. 507, Dickson's *History* II, is provisional solution, depending further on (F) $\Delta_f=(a^2-2b^2)(c^2-2d^2)\{(a^2-2b^2)$

$\cdot (c^2 - 2d^2) + 8abcd\} = \square$. Then $f/g = [(bc + ad)(ac - 2bd) \pm \Delta_f^{1/2}]/[(bc + ad)^2 - (ac + 2bd)^2]$; a/b and c/d are the same under a, b and $c, d \leftrightarrow f, g$. Thus each solution has 3 conjugates, each in turn having 2 new conjs., etc. ad inf. Erratum: in Euler (iii$_2$) the $+$ and $-$ signs should be interchanged in the formula for q, this solution being the same as Euler (i). Also Cunliffe's solution, p. 508, is the same as Euler (ii). (F) is satisfied trivially by $a, b, c, d, f, g = t, s, t, s, (t^2 - 2s^2)(t^2 \mp 2ts + 2s^2), \pm(t^4 + s^4)$; $2t, s, s, -t, \pm 2(2t^2 \mp 2ts + s^2), 2t^2 - s^2$, etc., such solution giving in general $x = y$ and $p = q$ and in particular the equilateral tet. Euler (iii$_2$) is self-conj. as to Δ_c, and $a, b, c, d, f, g = t, -s, t + 2s, t + s, (3t^2 + 8ts + 6s^2) \cdot (t^2 + 2s^2), -2(t^2 - 2s^2) \cdot (t^2 + 4ts + 2s^2)$ or 0. In Euler (ii) $a, b, c, d, f, g = t, s, t^4 - 2t^2s^2 + 4s^4, 2ts(t^2 - 2s^2), t^6 \mp t^5s - 8t^4s^2 \pm 14t^3s^3 \mp ts^5 + 8s^6, (s^2 - t^2) \cdot (t^4 \mp 4t^3s + 6t^2s^2 - 4s^4)$. There are also 9 Pythagorean parameters: $x, x_1 = k_x(t_x^2 \pm s_x^2); x_0 = 2k_xt_xs_x$, etc., such that $p = y_1 + y_0 = z_0 + z_1, q = x_0 + x_1 = z_0 - z_1, r = y_1 - y_0 = x_0 - x_1$. In Euler (iii$_2$) $k_x, t_x, s_x, k_y, t_y, s_y, k_z, t_z, s_z = 1, t^3 + 3t^2s + 2ts^2 - 2s^3, s(5t^2 + 8ts + 2s^3), 1, 2t^3 + 5t^2s + 4ts^2 + 2s^3, (t + s) \cdot (t^2 - 6s^2), t^2 - 2s^2, 2(t^2 + 2ts + 2s^2), t^2 + 4ts + 2s^2$; in Cunliffe's solution, $1, t(4t^4 + 8t^3s + 2t^2s^2 + s^4), 4t^4(t + s) + 2t^2s^2(t - s) - 3ts^4 - s^5, 1, (t + s) \cdot (4t^4 + 2t^2s^2 - s^4), t(4t^4 - 6t^2s^2 - 4ts^3 - s^4), 2t^2 - s^2, 2t(t + s)(2t^2 + s^2) + s^4, 2ts(2t^2 + 2ts + s^2)$. (Received March 10, 1954.)

377t. H. W. Becker: *Cunliffe's parametric solution for rational cuboids.*

In Dickson's *History* II, p. 499, Cunliffe[9] has a procedure 8 (or 4) times as prolific of solutions of $x^2 + y^2 = z^2, y^2 + z^2 = v^2, x^2 + z^2 = w^2$ as the Saunderson[2] (or Lenhart[12] = Rignaux[24c]). Cunliffe's forms C are of the 10th degree, but are the Euler[3] transforms of octics K: $u, x = \epsilon\zeta(\eta^2 \pm \theta^2); v, z = \eta\theta(\epsilon^2 \pm \zeta^2); y = 2\epsilon\zeta\eta\theta; \epsilon = \eta = 2r^3 - 2r^2s + 2rs^2 - s^3, \theta = 2(r + s)(r - s)^2, \zeta = (r - s)(2r^2 - 4rs + s^2); w, x = s(4r - 3s)(\gamma^2 \pm \delta^2); \gamma = (2r - s)(r^2 - s^2), \delta = r(2r^2 - 3rs + 2s^2)$. Cunliffe's forms cover number 3 of Kraitchik's table, Scripta Mathematica vol. XI (1945) p. 326; and the octics K cover numbers 1, 4, 5, 11, 13, and 22, reducing the number of mavericks to 36. The octics K have the 3 pairs of O'Riordan[33] parameters: $c = r^2 - s^2, d = r^2 - 4rs + 2s^2; e = 2r^2 - 4rs + s^2, f = 2c; a = c(2r - s)^2, b = r^2s(4r - 3s)$, whose conjugates are: $a' = (r^2 - s^2)(2r - 5s)^2, b' = s(4r - 3s) \cdot (3r - 4s)^2; e' = e(2r^4 + 4r^3s - 6r^2s^2 + s^4)^2, f' = f(2r^4 - 12r^3s + 14r^2s^2 - 6rs^3 + s^4)^2; c', d' = 12$-ics. The first O'Riordan conjugate OK is another octic, whose Pythagorean parameters are: $\gamma = (2r - 5s)(r^2 - s^2), \delta = \zeta = s(4r - 3s) \cdot (3r - 4s), \epsilon = (2r - 5s)(2r^2 - 3rs + 2s^2), \eta = 2(r + s) \cdot (r^2 - s^2), \theta = 2r^3 - 14r^2s + 22rs^2 - 11s^3$. The Pythagorean parameters are the same in any rational cuboid and its Euler transform. In all known basic cuboids, 2 or 3 of the component Pythagorean $\triangle$ are nonprimitive; the multipliers are the same in any cuboid and its O'Riordan conjugates. But in the Lebesgue transform (Kraitchik ibid. or Bull. Amer. Math. Soc. Abstract 59-6-559) only one $\triangle$ is nonprimitive. (Received March 10, 1954.)

378t. A. T. Brauer: *A note on the primitive roots* (mod p^n).

Maxfield and Maxfield [Bull. Amer. Math. Soc. vol. 58 (1952) p. 564] announced that they proved the existence of primitive roots (mod p^n) which are less than p. Without using this result it is proved in this paper that there exist at least $\phi(p - 1)/2$ primitive roots (mod p^n) which are less than p. If p is of the form $4k + 1$ and greater than 5, then there exist at least $3\phi(p - 1)/4$ such primitive roots. (Received March 10, 1954.)

379. A. T. Brauer (p) and H. T. LaBorde: *Numerical computation*

of the characteristic roots of a matrix and of the error in the approximate solution of linear equations.

Let A be a numerically given matrix with nonvanishing determinant. H. Wittmeyer [Zeitschrift für angewandte Mathematik und Mechanik vol. 16 (1936) pp. 287–300] published a method which gives estimates for the error in the approximate solution of a system of inhomogeneous linear equations with the coefficient matrix A. But this method can be used only if it is possible to determine a positive lower bound for the characteristic roots of the matrix A^*A where A^* is the conjugate transpose of A. In this paper a method for the determination of such a bound is obtained. For this purpose the results of Wittmeyer are extended to systems of homogeneous linear equations with vanishing determinant in order to obtain bounds for the components of a characteristic vector belonging to a characteristic root ω. It is not necessary that ω be numerically given. Using this result the characteristic roots of any numerically given matrix can be computed as exactly as necessary. (Received March 9, 1954.)

380*t*. J. L. Brenner and K. L. Cooke: *Hadamard matrices.*

An n by n matrix A with complex elements a_{ij} is said to have *dominant principal diagonal*, or to be an *Hadamard matrix*, if its elements satisfy the conditions $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ $(i = 1, \cdots, n)$. Such a matrix is known to be nonsingular, and the problem of delimiting the regions within which its characteristic roots must lie has been extensively discussed (cf. M. Parodi, *Mémorial des Sciences Mathématiques*, no. 118, Paris, Gauthier-Villars, 1952). M. Müller (Math. Zeit. vol. 51 (1948) p. 291) has pointed out the fact, of interest in discussing the location of the characteristic roots of an arbitrary nonsingular matrix, that if A is any nonsingular matrix, then there exists a nonsingular matrix B such that the matrix AB has dominant principal diagonal. In this article the authors show that B can always be taken to be the product of some permutation matrix by a triangular matrix T, and that, moreover, the first two (but in general not more than two) elements of the principal diagonal of T can be taken as 1. Furthermore, if one removes the restriction that two diagonal elements of T be 1's, T can always be chosen so that AB is a *triangular* matrix with dominant principal diagonal. (Received February 8, 1954.)

381. M. D. Burrow: *A generalization of the Young operator and its application to GL(2, q).*

In the first part of this paper the author proves a sufficient condition for the construction of a primitive idempotent of a group algebra from the linear representations of two subgroups. It is a generalization of the condition of J. von Neumann on which is based Young's method for the representations of the symmetric group. The new condition is not confined to the symmetric group. In addition it is shown that the condition is equivalent to the following: the representations induced in the group by the linear representations of the two subgroups have a single irreducible component in common, and neither induced representation contains this irreducible component more than once. The second part of the paper is an application to the group $GL(2, q)$. Complete bases for all the irreducible representations of this group of degrees 1, q, and $q+1$ are given explicitly. (Received March 8, 1954.)

382*t*. Leonard Carlitz: *Congruences for generalized Bell and Stirling numbers.*

If we put $E_0(x) = e^x - 1$, $E_k(x) = E_0(E_{k-1}(x))$ and set $E_0(uE_{s-1}(x)) = \sum_{r=1}^{\infty} (x^r/r!)$

$\cdot \sum_{c=1}^{r} u^{c} S(c, r, s)$, $E_{s}(x) = \sum_{r=1}^{\infty} x^{r} B(r, s)/s!$, then $S(c, r, s)$ and $B(r, s)$ are the generalized Stirling and Bell numbers in the notation of Becker and Riordan (Amer. J. Math. vol. 70 (1948) pp. 385–394). In their paper periodicity properties (mod p) of these numbers are obtained. In the present paper like results are obtained (mod p^{k}). Indeed the method applies to the coefficients of the functions $F_{0}(x)$, $F_{k}(x) = F_{k-1}(f_{k}(x))$, where each $f_{i}(x)$ satisfies a relation of the form $f'(x) = 1 + \sum_{1}^{\infty} a_{m} f^{m}(x)$ with integral a_{m}. (Received March 10, 1954.)

383t. Leonard Carlitz: *Some arithmetic properties of the Olivier functions.*

Let $k \geqq 1$ and put $\Phi_{0}(x) = \sum_{m=0}^{\infty} x^{mk}/(mk)!$; also set $(\Phi_{0}(x))^{-1} = \sum_{0}^{\infty} a_{mk} x^{mk}/(mk)!$. It is proved that the coefficients a_{m} satisfy congruences of Kummer's type. More generally the results apply to the coefficients of $(\Phi_{0}(x))^{-\lambda}$ as well as certain more general functions. (Received March 10, 1954.)

384t. Leonard Carlitz: *The coefficients of the reciprocal of $J_{0}(x)$.*

Put $f(z) = J_{0}(2x^{1/2}) = \sum_{0}^{\infty} (-1)^{m} z^{m}/(m!)^{2}$ and $(f(z))^{-1} = \sum_{0}^{\infty} \omega_{m} z^{m}/(m!)^{2}$. Congruences (mod p) are obtained for the ω_{m} and for the polynomials $\omega_{m}(x)$ defined by $f(xz)/f(z) = \sum_{0}^{\infty} \omega_{m}(x) z^{m}/(m!)^{2}$. The following more general result may be cited. Let $\{F_{m}\}$, $\{G_{m}\}$, $\{H_{m}\}$ be sets of polynomials with integral coefficients such that $H_{m} = \sum_{0}^{m} \binom{m}{r}^{2} F_{r} G_{m-r}$. Then if two of the sets of polynomials satisfy $F_{m} = F_{a_{0}} F_{a_{1}}^{p} F_{a_{2}}^{p^{2}} \cdots$ (mod p), where $m = \sum a_{i} p^{i}$, $0 \leqq a_{i} \leqq p-1$, the same is true of the third set. (Received March 10, 1954.)

385t. A. L. Foster: *On a unique subdirect factorization in universal algebras and their characterization by means of their identities.*

This communication extends such structure results as the following (Foster, *Generalized "Boolean" theory of universal algebras*, etc. Parts I and II, Math. Zeit. vols. 58, 59 (1953)), itself a generalization of familiar special cases (N. McCoy, G. Birkhoff, and others). *Theorem 1. Let A be a universal algebra which is functionally (strictly) complete. A sufficient (and necessary) condition for a universal algebra B to be isomorphic with a subdirect-power (=-sum or -multiple) of A is that B satisfy all identities satisfied by A.* Among further results we prove: *Theorem 2. Let $\widetilde{A} = \{A', A'', \cdots \}$ be an independent class of functionally complete algebras. A sufficient (and necessary) condition for a given algebra B to be expressible as a subdirect product of subdirect powers of a finite number of (distinct) $A^{(i_{1})}, A^{(i_{2})}, \cdots, A^{(i_{n})} \in \widetilde{A}$ is that B satisfy all identities common to $A^{(i_{1})}, \cdots, A^{(i_{n})}$. Theorem 3. In the foregoing notation, an algebra B which is representable in $\widetilde{A}$ as in Theorem 2, is uniquely so representable, i.e., the "primary divisors" $A^{(i_{1})}, \cdots, A^{(i_{n})}$ are unique in $\widetilde{A}$.* In Theorem 2 an *independent* class requires any two $A^{(i)} \in \widetilde{A}$ to be *cofunctional*, and the $A^{(i)} \in \widetilde{A}$ to be *uniformly coframal*. These concepts are too long for definition here. It is shown, however, that the conditions of Theorem 2 are widely realizable. (Received March 11, 1954.)

386t. Karl Goldberg: Log ($e^{x} e^{y}$) *in a free associative ring.*

Let x and y be elements of a free associative ring, and e^{x} and log $(1+x)$ be abbreviations for the formal power series which usually define these functions. Then let $z = \log (1+u)$ where $u = e^{x} e^{y} - 1$. The problem of computing the coefficients in the expression for z as a formal power series in x and y is due to Baker, Campbell, and Hausdorff, and is attacked in this paper. The primary result is a general formula for

the coefficients which can be applied easily to numerical computation. Simplifications of this formula are obtained for special cases including the result $z(s,\ t) = (-1)^t$ $\cdot [\sum_{i=1}^{s} \binom{s}{i} B_{s+t-i}]/s!t!$ where $z(s,\ t)$ is the coefficient of $x^s y^t$ in z, and B_k is the kth Bernoulli number in the usual notation. Preliminary studies of the computation of these coefficients were made on the National Bureau of Standards Eastern Automatic Computer. (Received February 22, 1954.)

387t. I. N. Herstein: *On the Lie ring of a division ring.*

If A is any associative ring, we can attach to it a Lie ring by introducing the new product $[a,\ b] = ab - ba$. If $[A,\ A]$ is the additive subgroup of A generated by all $xy -yx$ then $[A,\ A]$ is also a Lie ring under this product. It is proved that if A is a division ring of characteristic not 2, then any Lie ideal of $[A,\ A]$ must be contained in the center of A. (Received March 16, 1954.)

388t. D. G. Higman: *Modules with a group as operators.*

Let G be a group, S a subgroup of finite index. If M is a (unitary) G-module, M_S will denote the induced S-module. If m is a (unitary) S-module, $I_S(m,\ G)$ will denote the induced G-module. A G-module H will be called an extension of M if M is a G-submodule of H such that M_S is a direct summand of H_S. M will be called a quotient of H if there exists a G-submodule $\overline{M}$ of H such that $H/\overline{M}$ is isomorphic with M and M_S is a direct summand of H_S. The following generalizes theorems of W. Gaschütz (Math. Zeit. vol. 56 (1952) pp. 376–387) and B. Eckmann (Proc. Nat. Acad. Sci. U.S.A. vol. 39 (1953) pp. 35–42). For a G-module M, the following conditions imply each other: (a) M is a direct summand of $I_S(M_S,\ G)$, (b) M is a direct summand of every extension, (c) M is a direct summand of every G-module of which it is a quotient, (d) there exists an S-endomorphism α of M such that $\sum x\alpha x^{-1} = 1$, where the summation extends over a set of left representatives x for G over S. Corollary: Let G be a finite group, S a p-Sylow subgroup, and F a field of characteristic p. Then a representation submodule M of a representation module H for G in F is a direct summand of H if and only if M_S is a direct summand of H_S. It is also proved that if S is noncyclic, then G has indecomposable representations of arbitrarily high degree in F, which, combined with a result given in Abstract 59-4-339 enables one to state that G has cyclic p-Sylow subgroups if and only if the degrees of the indecomposable representations of G at characteristic p are bounded. (Received March 2, 1954.)

389t. D. G. Higman: *Induced and produced modules. I.*

Let S be a subring of an (associative) ring A. A module M will be called a (right) A-S-module if it has S as a ring of left and A as a ring of right operators such that $s \cdot ua = su \cdot a$ (s in S, u in M, a in A). To each (two-sided) S-module M there corresponds a pair $(I_S(M,\ A),\ \kappa)$ consisting of an A-S-module $I_S(M,\ A)$ and an S-homomorphism $\kappa : M \rightarrow I_S(M,\ A)$ determined uniquely up to an A-S-isomorphism by the property that for any A-S-module M' and any S-homomorphism $\delta : M \rightarrow M'$ there exists one and only one A-S-homomorphism $\overline{\delta} : I_S(M,\ A) \rightarrow M'$ such that $\delta = \kappa\overline{\delta}$; dually, there corresponds to M a pair $(P_S(A,\ M),\ \pi)$ consisting of an A-S-module $P_S(A,\ M)$ and an S-homomorphism $\pi : P_S(A,\ M) \rightarrow M$ determined uniquely up to an A-S-isomorphism by the property that for any A-S-module M' and any S-homomorphism $\delta : M' \rightarrow M$ there exists one and only one A-S-homomorphism $\overline{\delta} : M' \rightarrow P_S(A,\ M)$ such that $\delta = \overline{\delta}\kappa$. The existence is shown by exhibiting constructions. If A has an identity element 1 such that 1 is in S and 1 is the identity operator on the right of M then: $I_S(M,\ A) =$ the tensor product $M \otimes_S A$ turned into an A-S-module by

$s(u \otimes a) = su \otimes a$, $(u \otimes a)b = u \otimes ab$, with $\kappa : u \to u \otimes 1$; $P_S(A, M)$ the module of all additive homomorphisms $f : A \to M$ such that $f(as) = (fa)s$, turned into an A-S-module by $(f * a)b = f(ab)$, with $\pi : f \to f1$. $I_S(M, A)$ will be said to be induced by M if κ is an isomorphism; $P_S(A, M)$ will be said to be produced by M if π is onto. The former holds if and only if there exists an A-S-module generated by M; the dual condition is necessary and sufficient for the latter. (Received March 2, 1954.)

390t. D. G. Higman: *Induced and produced modules*. II.

The notation is that of the author's abstract immediately preceding. An A-S-module M will be called a quotient of (M', ϵ, σ), or a quotient of M', if M' is an A-S-module, ϵ an S-isomorphism of M into M', and σ an A-S-homomorphism of M' onto M such that $\epsilon\sigma = 1$; the concept of an extension of M is defined dually. If M is an A-S-module: $I_S(M, A)$ is induced by M and there exists one and only one A-S-homomorphism t of $I_S(M, A)$ onto M such that M is a quotient of $(I_S(M, A), \kappa, t)$; $P_S(A, M)$ is produced by M and there exists one and only one A-S-isomorphism $j : M \to P_S(A, M)$ such that $(P_S(A, M), \pi, j)$ is an extension of M. The following conditions imply each other: (a) M is a direct summand of $I_S(M, A)$, (b) M is a direct summand of every A-S-module of which it is a quotient; dually, the following conditions imply each other: (a') M is a direct summand of $P_S(A, M)$, (b') M is a direct summand of every extension. Suppose that A has an identity element 1, 1 in S, and that A has finite bases as a left and right S-module. If, as an S-module, $A[\cong I_S(S, A)]$ $\cong P_S(A, S)$—e.g., A a Frobenius algebra over a field S, A a finite group ring, S the subring spanned by a subgroup—then to every left S-basis $a_1, \cdots, a_n$ of A there corresponds a right S-basis $\bar{a}_1, \cdots, \bar{a}_n$ such that $\sum a_i \bar{a}_i$ is a center element of A, and $I_S(M, A) \cong P_S(A, M)$ for every S-module M having 1 as the identity operator on the right. In this case, the conditions (a), (b), (a'), (b') are all equivalent to ($*$) there exists an S-endomorphism α of M such that $\sum a_i \alpha a_i = 1$. This generalizes the author's theorem stated in Abstract 60-4-388. (Received March 2, 1954.)

391t. W. J. Klimczak: *A geometric solution of a system of linear inequalities*. Preliminary report.

This paper is concerned with a method of solutions of a system of linear inequalities of the form (A)$L_i(x) \equiv \sum_{j=1}^n a_{i,j} x_j < c_i$ ($i = 1, 2, \cdots, n$) which is analogous to the Gaussian elimination method for solving systems of linear equations. The $a_{i,j}$ and c_i ($i, j = 1, 2, \cdots, n$) are known real constants and the solution of (A) is defined to be the totality of points $x = (x_1, x_2, \cdots, x_n)$ in n-dimensional Euclidean space E_n for which (A) is satisfied. Let H_i be the hyperplane in E_n which has $L_i(x) = c_i$ as its equation, and let R_i be the set of all points x of E_n such that $L_i(x) < c_i$. The method of solution of (A) consists of applying a sequence of n linear, nonsingular transformations to E_n. Let T_i ($i = 1, 2, \cdots, n$) denote the matrices of these transformations and $x^{(n)} = xT_1T_2 \cdots T_n$, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)})$. The sequence consists of n-dimensional shears and is defined so that it transforms H_i into the hyperplane $H_i^{(n)}$ such that $H_i^{(n)}$ is orthogonal to the $x_i^{(n)}$-axis in the $x_1^{(n)} x_2^{(n)} \cdots x_n^{(n)}$-system. (A) is then reduced to a simpler system of the form (B) $f(a_{i,i}) x_i^{(n)} < c_i$ ($i, j = 1, 2, \cdots, n$), each inequality of which defines a set of points $R_i^{(n)}$, the transform of R_i. The inverse image of the intersection of the point sets $R_i^{(n)}$ ($i = 1, 2, \cdots, n$) is the solution of (A). Thus the solution of (A) is given by $x = x^{(n)} T_n^{-1} T_{n-1}^{-1} \cdots T_1^{-1}$, where $x^{(n)}$ is determined by (B). The method may be used to solve the system of equations $L_i(x) = c_i$ ($i = 1, 2, \cdots, n$). (Received March 9, 1954.)

392t. Joachim Lambek: *A necessary condition for arithmetic equivalence between character algebra and center of group ring.*

Let G be a finite group, Z the center of its group algebra over an algebraically closed field of characteristic 0, H its character algebra over the same field (i.e. the basis elements of H are multiplied like the characters of G). It is known that Z and H have the same number s of basis elements, and that their multiplication tables consist of rational integers. When are they arithmetically equivalent, in the sense that the subrings of linear combinations of the basis elements with algebraic or rational integers as coefficients are isomorphic? All isomorphisms between Z and H are constructed explicitly in matrix form. Arithmetic equivalence requires that one of these matrices be unimodular. This becomes: $\prod_{i=1}^{s} h_i = \prod_{j=1}^{s} n_j$, where the h_i are the numbers of elements in the classes of conjugate elements, and the n_j are the degrees of the irreducible representations of G. The condition fails e.g. for the quaternion group of order 8. Does it imply that G is abelian? (Received March 10, 1954.)

393. Karel deLeeuw: *Applications of relative cohomology theory to algebraic number theory.*

This paper takes as point of departure notes of E. Artin (*Algebraic numbers and algebraic functions* II, to appear). There, for every finite group G and G-module A, cohomology groups $H^n(G, A)$ are defined for all integers n. It is proved that if A is the multiplicative group of a local field or is idele classes there is a canonical isomorphism $H^n(G, A) \cong H^{n-2}(G, Z)$ where Z is the integers with trivial action. In this paper, for every subgroup H of G, two kinds of relative cohomology groups are defined and two exact sequences constructed, one containing restriction and the other verlagerung. If A is as above there are canonical isomorphisms between the sequences for A and the sequences for Z which shift the dimension by two. In particular, the relative cohomology of A depends only on G and H and is independent of the arithmetic. In a number field the structure of the quotient group of norms of ideles module norms, in a layer that is not necessarily normal, can be determined in terms of the relative cohomology groups and is found to depend only on the decomposition groups and is otherwise independent of the arithmetic. (Received March 8, 1954.)

394. Arthur Mattuck: *Abelian varieties over p-adic fields.*

Let k be a field of characteristic 0 complete under a non-archimedean absolute value, and let A be an abelian variety of dimension d defined over k. That is, A is a d-dimensional algebraic variety embedded in a projective space over some universal domain containing k, whose points form without exception an algebraic group; k is to be a field of definition both for A and the group composition law. It is proved that the group of points on A rational with respect to k contains a subgroup isomorphic and homeomorphic to the direct sum of d copies of the integers of k. Further, if k is a finite algebraic extension of the rational p-adic field, then this subgroup is of finite index. An identical pair of statements may be made about the group of rational divisor classes of degree 0 of an algebraic function field in one variable over k as constant field: this follows from Chow's construction of an abelian variety defined over k (the Jacobi variety) whose rational points represent these divisor classes in a 1-1 fashion. In this form the case genus 1 was proved by Lutz (J. Reine Angew. Math. vol. 177 (1937)); the general statement was conjectured by Weil. (Received March 9, 1954.)

395t. E. V. Schenkman: *The existence of outer automorphisms of some nilpotent groups of class 2.*

Let G be a p-group whose commutator subgroup is in the center and such that for some fixed number p^k the p^kth powers of all the elements of G are in the commutator subgroup. Then the following two statements are valid: (1) G has an outer automorphism (except if G is of order 2). (2) If the order of G is odd, then it divides the order of the automorphism group of G provided G is not Abelian. (Received March 23, 1954.)

396. Abe Shenitzer: *Decomposition of a group with a single defining relation into a free product.*

Let $G(a_1, \cdots, a_n; R(a_1, \cdots, a_n) = 1)$ be a group with a single defining relation. Using theorems by Grushko (A. G. Kurosh, *Theory of groups*, Gostekhizdat, 1944) and J. H. G. Whitehead (Ann. of Math. vol. 37 (1936)) it is shown that G cannot be decomposed into a free product if (a) R is a product $R_1 \cdots R_k$ of words $R_1, \cdots, R_k$, each of which is minimal (T) (in the sense of Whitehead) and of length greater than 1, and the sets $S_1, \cdots, S_k$ of generators of G appearing in $R_1, \cdots, R_k$, respectively, are pairwise disjoint or (b) all the exponents of the generators of G appearing in R are greater than or equal to 2. These statements are derived from the theorem that G is decomposable into a free product if and only if there exists a form of R which is minimal (T) and which contains at most $n-1$ of the n generators of G. (Received March 9, 1954.)

397. Morris Weisfeld: *A note on purely inseparable extensions.*

The results of Baer (Sitzungsberichte Heidelberger Akademie (1927) pp. 15–32) are generalized to purely inseparable extensions of finite exponent by means of the concept of a higher derivation due to Hasse-F. K. Schmidt (J. Reine Angew. Math. vol. 177 (1937) pp. 215–237). The main result shows that given a purely inseparable extension K of exponent e over a field C, there exists a higher derivation of rank less than or equal to p^e in K having C as the subfield of constants. (Received March 11, 1954.)

398. A. B. Willcox: *On the group algebra of the direct product of a compact group and a locally compact abelian group.*

Two properties of the group algebra $L^1(G)$ of the direct product G of a compact group and a locally compact abelian group are proved in this note. (1) The structure space of maximal regular ideals of $L^1(G)$ is Hausdorff and so is the structure space of the algebra formed by formally adjoining an identity to $L^1(G)$. The proof is based on results due to Kaplansky [Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949)] and allows techniques due to Silov [Trav. Inst. Math. Stekloff, 1947] and the author [Doctoral dissertation, Yale University, 1953] to be used in studying these algebras. (2) $L^1(G)$ is a cross algebra (in the sense of Schatten [Annals of Mathematics Studies, no 26, 1950] with the obvious extension of the multiplication operation) of the group algebras of the component groups of G (here it is sufficient that the component groups be locally compact). (1) and (2) are used to show that the Wiener Tauberian theorem in its ideal theoretic form holds for this case. The note concludes with one further application of a theorem from the above mentioned doctoral dissertation. The result concerns the spectral resolution of a closed ideal in $L^1(G)$ and says, in part, that if

$k(h(I))$, the kernel of the hull of I, is a regular ideal and if the kth derived set of $h(I)$ is void, then $[k(h(I))]^k \subset I \subset k(h(I))$. (Received March 11, 1954.)

ANALYSIS

399. W. G. Bade: *A theorem on convergence of projections.*

Let X be a reflexive Banach space and let B be a bounded Boolean algebra of projections in X, i.e. $|E| \leq M$, $E \in B$. Theorem: The strong and weak operator topologies coincide on B. The theorem is proved by reducing the problem to an integration problem with respect to a vector valued measure on a suitable measure space. (Received March 10, 1954.)

400. J. H. Barrett: *Oscillation and boundedness of solutions of* $y'' + q(x)y = 0$.

This paper is concerned with a modified polar coordinate transformation of the second order differential equation $y'' + q(x)y = 0$, specifically, $y = r(x) \sin \theta(x)$, $y' = w(x)r(x) \cos \theta(x)$. This transformation with $w(x) = 1$ was introduced by Prüfer. Certain conditions for oscillation, nonoscillation, and boundedness of solutions involving $w(x)$ are given. It is noted that certain choices of $w(x)$ give known results and that other choices give new results. Among the new results are: Theorem. If $\int_a^x |q| = O(x)$ as $x \to \infty$; $w(x) > 0$, $w'(x) \leq 0$, and $Q(x) = \int_a^x |w - q/w|$ for $x \geq a$, then $\lim \sup_{x \to \infty} (|y(x)|/w(x)) \exp Q(x)/2 > 0$. This theorem is proved by applying techniques of Levinson (Duke Math. J. vol. 8 (1941) pp. 1–10) to inequalities obtained by this transformation and a specialization of w gives a result of Levinson's as a corollary. Theorem: If there exists $w(x) > 0$ such that $w'(x) \leq 0$, $w^2(x) \leq q(x)$ and $\lim_{x \to \infty} [\int_a^x w + \ln w] = \infty$, then all solutions are oscillatory. (Received March 10, 1954.)

401t. Stefan Bergman: *On functions of two complex variables in domains with a distinguished boundary surface.*

Generalizing the investigations on analytic functions of two complex variables in Math. Zeit. vol. 36, p. 177 and vol. 39, p. 76, the author considers domains which are bounded by n analytic hypersurfaces h_S^3. The sum I^2 of intersections $i_{KS}^2 = h_K^3 \cap h_S^3$, $K \neq S$, S, $K = 1, 2, \cdots, n$, forms the distinguished boundary surface of M^4. The author introduces functions $\phi_S(Z)$, $Z = (z_1, z_2)$, which are orthonormal in M^4, when integrating over the distinguished boundary surface I^2, and shows that $\sum_{S=1}^{\infty} |\phi_S(Z)|^2 < \infty$, for $Z \in M^4$. The proof uses the lemma stating that the minimum of $\sum_{K,S} \int_{i_{KS}^2} |f(z_1, z_2)|^2 [\partial(z_1, z_2)/\partial(\lambda_K, \lambda_S)] d\lambda_K d\lambda_S$ under the condition $f(t_1, t_2) = 1$, $(t_1, t_2) \in M^4$, is uniformly bounded in every closed subdomain of M^4. Here $\{f\}$ denotes the totality of functions which are regular in M^4, continuous in M^4. Using this result the author obtains a development in a domain M^4 in terms of values on the distinguished boundary surface. In the case of three (and more) variables one obtains functions which are orthonormal when integrating over three- and four-dimensional boundary manifolds. Corresponding considerations lead to analogous representations. Finally, the specialization of the considerations to the case of functions satisfying linear differential equations is discussed. (Received April 8, 1954.)

402. P. L. Butzer: *Summability of generalized Bernstein polynomials.*

Let $f(x) \in L[0, 1]$, then the generalized Bernstein polynomials defined as $P_n'(x)$

$= \sum_{\nu=0}^{n} (n+1)\binom{n}{\nu} x^{\nu}(1-x)^{n-\nu} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} f(t) dt$ have the property that $\lim_{n\to\infty} P'_n(x)$ $= f(x)$a.e. in $[0, 1]$. In this note the polynomials $\mathfrak{P}'_n(x) \equiv (n+1)P'_n(x) - nP'_{n-1}(x)$ are considered. It is shown if $f(x)$ is defined on $[0, 1]$, $|f(x)| \leq M$, the sequence $\{P'_n(x)\}$ is summable $(C, -1)$ to $f(x)$, i.e. $\mathfrak{P}'_n(x) \to f(x)$ at every point $x \in [0, 1]$ where $f''(x)$ exists. If $f(x)$ has a modulus of continuity $\omega(\delta)$ in $0 \leq x \leq 1$, with $\omega(\delta) = o(\delta^{1/2})$, then $\lim_{n\to\infty} \mathfrak{P}'_n(x) = f(x)$ uniformly in every interval $[\epsilon, 1-\epsilon]$, $0 < \epsilon < 1/2$. Furthermore, there exist functions continuous on $[0, 1]$ such that the $\mathfrak{P}'_n(x)$ diverge at a given point x, $0 < x < 1$. The proofs of the latter two results depend on the Lebesgue constants for the $\mathfrak{P}'_n(x)$ which are found to be of order $O(n^{1/2})$. (Received March 8, 1954.)

403. R. H. Cameron (p) and J. M. Shapiro: *Nonlinear integral equations.*

Under suitable conditions on y and F, this paper gives various expressions for the solution $x(t)$ of the nonlinear integral equation $y(t) = x(t) + \int_0^t F(t, s, x(s)) ds$ in terms of Wiener integrals involving F and y. These results generalize the main results of two papers by Cameron and Martin [Amer. J. Math. vol. 66 (1944) pp. 281–298, and Ann. of Math. vol. 51 (1950) pp. 629–642] and a paper of Cameron, Lindgren, and Martin [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 138–143]. This research was sponsored by the Office of Ordnance Research, U. S. Army, under contract DA-11-022-ORD-489. (Received March 8, 1954.)

404t. Y. W. Chen: *On a problem of minimal surface with polygonal boundary.*

Let C be a simple closed convex polygon on the x, y plane. The problem is to find a minimal surface solution $z = z(x, y)$ with the following properties: (1) it extends in the exterior of C to infinity, (2) $\partial z/\partial n = 0$ on C, (3) at infinity $z_x : z_y : -1 = \beta : 0 : -1$. This problem can be interpreted as a problem of disturbed uniform stream around profile C with given velocity at infinity—using Chaplygin's approximate adiabatic law. The present paper is concerned with the existence of solution and its behavior around the vertices of the polygon. It is known that such a solution can *not* exist, if around the corners the solution behaves in the same manner as in the case of incompressible flow. The existence proof is based on an earlier work of the author (Trans. Amer. Math. Soc. vol. 65), in which minimal surface solutions were obtained in parametric representation by harmonic functions $x(w)$, $y(w)$, and $z(w)$, with $w = u + iv$ and $|w| \leq 1$. A solution $z = z(x, y)$ is obtained if it is shown that the Jacobian $\partial(x, y)/\partial(u, v) \neq 0$. The significant feature in the proof is that the boundary mapping between $|w| = 1$ and C is not a one to one point transformation. Except for one or two corners, every other corner has as its antecedent a whole arc on $|w| = 1$. The function $z(x, y)$ is therefore discontinuous at these corners, although the parametric representation is analytic and continuous for $|w| \leq 1$. For applications the solution can be represented explicitly by a formula which generalizes the classical formula of Schwarz and Christoffel. (Received March 8, 1954.)

405. Ruth M. Davis: *The regular Cauchy problem for the Euler-Poisson-Darboux equation.*

Let $u(x, t)$ denote $u(x_1, x_2, \cdots, x_m, t)$ and let k be a real parameter. A solution to the equation (1) $L[u] = u_{tt} + kt^{-1}u_t - \Delta u = 0$ with $u(x, t_1) = f(x)$, $u_t(x, t_1) = 0$ is found for $t_1 > 0$ by a modified Riesz method. A function (2) $V^{\alpha}(x, t_0; y, t) = As^{\alpha-m-1}(t_1/t)^{k/2}$ $\times F[(1-k/2), k/2; (\alpha+1-m)/2; (-s^2/4tt_1)]$, where A is known explicitly, is ob-

tained, for which $M[V^{\alpha+2}] = V^{\alpha}$ where $M[u]$ is the adjoint operator to $L[u]$. F denotes the hypergeometric series. Applying Green's Theorem to u and $V^{\alpha+2}$, it is found by a limiting process as $\alpha \to 0$ that the solution to (1) is (3) $u(x, t) = a \sum_{j=0}^{\infty} [b_j t^{-k/2-i} t_1^{k/2-j-1} \cdot I^{(2j+3-m)/2} F(s) + c_j t^{-k/2-i} t_1^{k/2-j} s^{1/2} I^{(2j+1-m)/2} F(s)]$ where the coefficients are explicitly known. $I^\beta F(s)$ is the Riemann-Liouville integral with $F(s) = M(x, s^{1/2}; f) s^{(m-2)/2}$ where $s = (t-t_1)^2$ and $M(x, s^{1/2}; f)$ denotes Poisson's mean value. (Received February 11, 1954.)

406. Albert Edrei: *On the zeros of successive derivatives.*

A recent result of the author [Bull. Amer. Math. Soc. Abstract 60-2-221] yields a number of propositions relative to the limit points of the zeros of the derivatives of an analytic function. Three of the simplest will be stated. (I) If the entire function $f(z)$ is real for real values of z and if it is bounded on the real axis, then every real point x_0 for which $\lim_{n \to \infty} \{f^{(n+1)}(x_0)/f^{(n)}(x_0)\} = \infty$ is a point of accumulation of zeros of the derivatives of $f(z)$. (II) Consider the function $f(z) = \sum a_v z^v$ and assume that $f(z)$ is not an entire function of exponential type. Then it is possible to choose a sequence of numbers ω_v, each of which is equal to $+1$ or -1, and such that the origin is a limit point of the zeros of the derivatives of $h(z) = \sum a_v \omega_v z^v$. (III) The zeros of the derivatives of Mittag-Leffler's function $\sum z^n/\Gamma(1+n/2)$ have no finite limit point. The latter proposition settles a question of Pólya [Bull. Amer. Math. Soc. vol. 49 (1943) p. 181, footnote 2]. (Received March 8, 1954.)

407t Herbert Federer: *An analytic characterization of distributions whose partial derivatives are representable by measures.*

All first order partial derivatives of a distribution (in the sense of Laurent Schwartz) in Euclidean n-space are representable by locally finite signed Borel measures if and only if the distribution itself is representable by a locally Lebesgue integrable function f such that, for $j = 1, 2, \cdots, n$ and each bounded interval I, the essential variation of $f(y_1, \cdots, y_{j-1}, t, y_j, \cdots, y_{n-1})$ with respect to t on I is locally Lebesgue integrable with respect to y in $(n-1)$-space. (The essential variation of a real-valued Lebesgue measurable function g on an interval I is the supremum of $\sum_{k=1}^{p} |g(t_k) - g(t_{k-1})|$ where $t_0 < t_1 < \cdots < t_p$ are points of approximate continuity of g in I.) (Received March 8, 1954.)

408t. R. F. Gabriel: *The Schwarzian derivative and convex functions.*

Let $f(z) = 1/z + a_1 z + a_2 z^2 + \cdots$ be regular in $0 < |z| < 1$. Let c be the smallest positive root of the equation $2x^{1/2} - \tan(x^{1/2}) = 0$. If $\{f(z), f\}$ denotes the Schwarzian derivative of $f(z)$ and if $|\{f(z), f\}| \leq 2c$ for $|z| < 1$, then $f(z)$ is univalent in $0 < |z| < 1$ and maps the interior of each circle $|z| = r < 1$ onto the exterior of a convex region. The constant c is a best possible one. Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ be regular in $|z| < 1$ and real on the real axis. Let $\mathcal{R}\{zg'(z), z\} \geq -\pi^2/2$ in $|z| < 1$. Then $g(z)$ is univalent in $|z| < 1$ and maps the interior of the unit circle into a region which is convex in the direction of the imaginary axis. The constant $-\pi^2/2$ is a best possible one. The results are obtained by a modification of the method employed by Nehari [Bull. Amer. Math. Soc. vol. 55 (1949) pp. 545-551]. (Received March 4, 1954.)

409. Dieter Gaier: *On the equivalence of the two methods of Borel summability.*

If $B(x, s_n)$ and $B'(x, s_n)$ are Borel's exponential and integral transform, respec-

tively, of a series $\sum a_r$, $s_n = \sum_{r=0}^{n} a_r$, then $B(x, s_n) \to s$ implies $B'(x, s_n) \to s$ $(x \to +\infty)$. The converse is not true in general; however, the author shows its validity under the assumption (1) $a_n = O(k^n)$ $(n \to \infty$, k any fixed constant), thus improving conditions by Garten $[a_n = O(n^k)]$ and Karamata $[a_n = O(e^{n\rho})$ $(n \to \infty, \rho < 1/3)]$. The proof uses a simple fact about integral functions $f(z)$ of mean type of order one, namely that $f(x) \to s$ $(z = x + iy)$ implies $f'(x) \to 0$ $(x \to +\infty)$. The condition (1) cannot be improved to $a_n = O(k^n \cdot n^{\epsilon n})$ $(\epsilon > 0$, fixed). There are close relations to the problem, when the change of index for Borel-summable series is allowed [cf. Math. Zeit. vol. 58 (1953) pp. 453–455]. (Received March 2, 1954.)

410. R. E. Gomory: *Critical points at infinity and forced oscillation*.

Let $dx/dt = X(x, y)$, $dy/dt = Y(x, y)$ be a pair of ordinary differential equations. Let s be the sum of the indices of the critical points. Let $X(x, y)$, $Y(x, y)$ be polynomials such that when the equations are extended to the projective plane, (1) there are critical points on the line at infinity $(z = 0)$ and they are simple, (2) there are no two consecutive saddle points on $z = 0$, (3) $s \neq 0$. It is then shown that the equations $dx/dt = X(x, y) + e_1(t)$, $dy/dt = Y(x, y) + e_2(t)$, with $e_1(t)$ and $e_2(t)$ periodic of period T, have a periodic solution. The first order systems which arise from $d^2x/dt^2 + f(x)dx/dt + g(x) = 0$ by setting $dx/dt = y$ generally have complicated critical points at infinity. For these systems, with f and g polynomials, and degree $f \geq$ degree $g > 0$, the critical points on $z = 0$ are completely analyzed, and their nature found to depend only on the leading coefficients a_n and b_m of f and g. Geometric information, such as the existence of limit-cycles in equations like that of van der Pol, is then easily deduced. Finally forced oscillations are shown to exist for the equation $d^2x/dt^2 + f(x)dx/dt + g(x) = e(t)$, $e(t)$ periodic, provided that (1) degree $f \geq$ degree g, (2) degree g is odd; (3) not both $b_m > 0$ and degree f odd hold. Both forced oscillation theorems show the existence of periodic motions for systems not necessarily dissipative for large displacements. (Received March 10, 1954.)

411t. E. C. Gras: *On a generalization of the integrals of Weber and Schafheitlin*.

The following formula generalizing the theory of Weber's integrals is obtained: $\lim y^{-jb-k} \int_0^\infty e^{-xt} J_{r+j}(yt) J_{r+k}(bt) t^{-(1+j+k)} dt = \beta \sum_{i=0}^{j} C_i (b/y)^{2i+r} + (\pi - \beta) \sum_{i=0}^{k} D_i (y/b)^{2i+r}$, where $x \to 0^+$, and $x^{-1}(y - b) \to \cot \beta$, and $r > 0$, $1 + j + k \geq 0$, and $C_i = 0$ if $j < 0$, and $D_i = 0$ if $k < 0$. The proof is based on an extension of the generalized axially symmetric potential theory of A. Weinstein (Bull. Amer. Math. Soc. vol. 59 (1953) pp. 20–38). The result includes as a special case the integrals of Weber-Schafheitlin. (Received February 10, 1954.)

412. Sigurdur Helgason: *Banach algebras of almost periodic functions*.

The space of continuous almost periodic functions $f(x) \sim \sum a(\lambda) e^{i\lambda x}$ is considered as a Banach algebra A under convolution-multiplication. The automorphisms T of A are determined, and correspond by $Tf(x) \sim \sum a(\sigma(\lambda)) e^{i\lambda x}$ to a group H of permutations of R. H contains all permutations of the form $\sigma(\lambda + \nu) = \sigma(\lambda) + \sigma(\nu) - \sigma(0)$, and these permutations correspond exactly to the isometric automorphisms of A. A function $q(\lambda)$ is called a multiplier if $\sum q(\lambda) a(\lambda) e^{i\lambda x}$ is an almost periodic Fourier series whenever $\sum a(\lambda) e^{i\lambda x}$ is. The multipliers are precisely the functions of the form $q(\lambda) = q_1(\lambda) - q_2(\lambda) + iq_3(\lambda) - iq_4(\lambda)$ where the $q_i(\lambda)$ are positive definite (in general

nonmeasurable) functions. (The continuous multipliers were determined by R. Doss, Ann. of Math. vol. 46 (1945).) A multiplier $q(\lambda)$ induces an operator on A which is isometric if and only if $q(\lambda+\nu)q(0) = q(\lambda)q(\nu)$, $|q(0)| = 1$. For the special case where $q(\lambda)$ is the characteristic function of a set $S \neq R$ the following is proved: (1) S has "mean density" 0. (2) If $f_S(x) \sim \sum_{\lambda \in S}(\lambda)e^{i\lambda x}$ is positive whenever $f(x)$ is, then S is a subgroup of R, and all subgroups have this property. The results hold for all abelian groups G with exception of (1) which is proved under the assumption that the character group $\widehat{G}$ is an infinitely divisible group. (Received March 9, 1954.)

413t. Sigurdur Helgason: *On the distribution of values of analytic almost periodic functions.*

A normal almost periodic function is a nonconstant analytic almost periodic function whose Laurent-Dirichlet expansion contains a smallest or a largest exponent. The inverse of such a function exists and is again of this type (H. Bohr, Ann. of Math. vol. 32 (1931) pp. 247–260), and the same is proved for the composition of two such functions. The main results of Nevanlinna's theory for the distribution of values of single-valued analytic functions are extended to normal almost periodic functions. An essential tool is the generalization of Jensen's formula, due to B. Jessen (Math. Ann. vol. 108 (1933) pp. 485–516). The characteristic function $T(\sigma)$, the defect $\delta(a)$, and the ramification index $\theta(a)$ can be introduced for any complex number a. $T(\sigma)$ is increasing and convex and describes the asymptotic behavior of the corresponding normal almost periodic function $f(s)$. Two different cases arise: (1) $f(s)$ defined in the whole s-plane. (2) $f(s)$ defined in a half plane, say $(-\infty, 0)$. In the first case $\lim_{\sigma \to \infty} T(\sigma) = \infty$ and $\sum_a(\delta(a)+\theta(a)) \leq 1$, and in the second case the quantity $\kappa = \lim \sup_{\sigma \to 0} 1/(1-e^{\sigma})$ $\cdot T(\sigma)$ is introduced, and the relation $\sum_a(\delta(a)+\theta(a)) \leq 1+2\kappa$ is proved. In particular, if $\kappa = 0$, $f(s)$ omits at most one value. (Received March 9, 1954.)

414. Peter Henrici: *On certain series expansions involving Whittaker functions and Jacobi polynomials.*

Substitution of polar coordinates in (*) $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + (\mu/x)\partial u/\partial x + (\nu/y)\partial u/\partial y + [\lambda - k^2(x^2+y^2)]u = 0$ leads to a set of solutions F_n of (*) expressible in terms of Whittaker functions and Jacobi polynomials. With a view towards deriving some functional relations involving hypergeometric functions, a technique is developed to construct expansions of functions of a certain class E of regular solutions of (*) in terms of the F_n. The method (related to Bergman's integral operator method) consists in setting up a 1-1 correspondence between E and the class of even analytic functions of one complex variable regular at 0 by associating to a $u(x, y) \in E$ the function $u(z/2, -iz/2)$. Since F_n is shown to correspond to z^{2n}, the expansion problem in question reduces to the determination of the Taylor expansion of an analytic function of one variable. Application of this technique to some special solutions of (*) gives rise to three expansions involving various types of hypergeometric functions. As special cases and corollaries one obtains, among others, Bateman's addition theorem for Bessel functions, Ramanujan's formula for the product of two $_1F_1$, Erdélyi's addition theorem for the product of two M-functions, and Bailey's decomposition formula for Appell's function F_4. (Received March 9, 1953.)

415. W. B. Jurkat: *Elementary Mercerian theorems.*

A Mercerian theorem is generally spoken of as an equivalence theorem between two summability methods, one of them usually is the ordinary convergence. A and B

being regular and normal matrices, I the unit matrix, then there are considered the following Mercerian theorems: $rA + (1-r)I \approx I$ or more general $rA + (1-r)B \approx B$, $0 < r < 1$. In the first case it is possible to give necessary and sufficient conditions introducing the conception of a mean value theorem for A. Several monotone conditions on the elements of the matrix A, sufficient for the mean value theorem, are established, and each of these conditions leads to a Mercerian theorem of the first kind. In the more general second case similar sufficient conditions can be given. They mean roughly speaking that A should be stronger than B. Some known and also some unknown special cases are included in the theorems established. A special case covered is $rC_k + (1-r)C_i \approx C_i$ with $0 < r < 1, 0 \leq i \leq k \leq 1$, where C_i, C_k denote Cesàro methods of orders i, k respectively. (Received March 9, 1954.)

416t. Viktors Linis: *Power series in fields with nonarchimedean valuation.* Preliminary report.

The case when the valuation is nonarchimedean and nondiscrete is studied. It is shown that for functions defined by convergent power series the maximum modulus principle and Cauchy estimate holds. For entire functions a factorization theorem analogous to that of Weierstrass in complex analysis is established. The proofs are based on the theorems on moduli of polynomials and distribution of their zeros. (Received March 8, 1954.)

417t. C. N. Moore: *On questions of regularity for Nörlund means of double series.*

Suppose that the matrix p_{mn} for the Nörlund mean of a double series yields a regular method. The question arises as to whether or not the individual rows and columns of this matrix yield regular methods for simple series. It can be shown by examples that they do not. Furthermore, it can also be shown by examples that in case the individual rows and columns do not furnish a regular method for simple series, the corresponding double Nörlund means are not necessarily consistent. Hence the requirement of regularity for the rows and columns is an essential supplementary condition in extending Nörlund's theorem to double series. (Received March 8, 1954.)

418. Jürgen Moser: *A singular problem of perturbation theory.*

Consider an eigenvalue problem $(P + \epsilon Q)u(x) = \lambda u(x)$, where P, Q are differential operators with respect to x of even orders $2p$ and $2q$, resp. In $a \leq x \leq b$ the coefficients of P, Q are assumed to be real and infinitely differentiable. The coefficients of the highest derivative in $(-1)^p P$ and $(-1)^q Q$ are positive. At $x = a$, $x = b$ the solution has to satisfy q boundary conditions. The essential assumption is $p < q$; hence the order of $P + \epsilon Q$ changes for $\epsilon = 0$. The problem is to investigate the dependence of the eigenvalues on the positive parameter ϵ, especially for $\epsilon \to 0$. Under general assumptions it can be proved that the asymptotic behavior of an eigenvalue $\lambda(\epsilon)$ for $\epsilon \to 0$ is given by an asymptotic series $\lambda(\epsilon) \sim \lambda_0 + \eta \lambda_1 + \eta^2 \lambda_2 + \cdots$, where $\eta = \epsilon^{1/(2q-2p)}$. In general the series diverges, but approximates $\lambda(\epsilon)$ in the sense of Poincaré. Also the eigenfunctions can be described by such asymptotic series. The essential difficulty of the problem lies in the fact that "unperturbed" eigenvalue problem, for $\epsilon = 0$, is not completely known. Namely, for $\epsilon = 0$ there are too many boundary conditions prescribed. It can be decided (from pure algebraic arguments) which boundary conditions have to be cancelled for $\epsilon = 0$. The proof uses the known results concerning the asymptotic behavior of solutions of differential equations containing a small parameter in the highest derivative. (Received March 9, 1954.)

419t. Leopoldo Nachbin: *Bornological spaces of continuous functions and cartesian products.*

The vector space of all real-valued continuous functions on a completely regular space T is bornological in the sense of Mackey (cf. Bourbaki, Annales Inst. Fourier vol. 2 (1950) pp. 5–16) with respect to the compact-open topology if and only if T is saturated, i.e. complete under the weakest uniform structure rendering uniformly continuous all real-valued continuous functions on T. Every regular space in which any open covering contains a countable subcovering is saturated. A discrete space is saturated if and only if there is no Ulam measure on it. Let $E = \prod_{i\in I}E_i$, $E_i \neq 0$, be a cartesian product of real vector spaces and, for $J \subset I$, let E_J be the canonical image in E of $\prod_{i\in J}E_j$. Then every convex subset $V \subset E$, containing 0 and absorbing any set $\prod_{i\in I}F_i$ whose components $F_i \subset E_i$ are finite, must contain some E_J with $I - J$ finite if and only if the discrete space I is saturated. It results in the known fact that E is bornological if and only if every E_i is bornological and I is saturated. (Received January 21, 1954.)

420. Alexander Peyerimhoff: *Summability factors for absolute Cesàro summable series.*

By a well known theorem of I. Schur, necessary and sufficient conditions for numbers ϵ_n in order to be (C_α, C_β) summability factors, i.e. to transform every C_α-summable series $\sum a_n$ into a C_β-summable series $\sum a_n\epsilon_n$ ($\alpha \geq 0$, $\beta \geq 0$), are the following: $\sum \binom{n+\alpha}{n}|\Delta^{\alpha+1}\epsilon_n| < \infty$, $\epsilon_n = O(n^{\beta-\alpha})$. The report deals with the corresponding problem for absolute Cesàro summability and proves for $(|C_\alpha|, |C_\beta|)$ summability factors the necessary and sufficient conditions $\Delta^\alpha\epsilon_n = O(1/n^\alpha)$, $\epsilon_n = O(n^{\beta-\alpha})$. This theorem generalizes some results of Bosanquet and Kogbetliantz. (Received March 9, 1954.)

421. P. C. Rosenbloom: *Singularities of solutions of nonlinear hyperbolic equations.* Preliminary report.

We show that if $u_{tt} - u_{xx} = u^2$ for $|x| \leq T - t$, $0 \leq t \leq t_0 \leq T$, $u(x, 0) = u_0(x)$, $u_t(x, 0) = 0$ for $|x| \leq T$, and if $u_0(x) \geq a > (K/T)^2$, where K is a certain numerical constant, then $t_0 < T$, i.e., $u(0, t)$ has a singularity in the interval $[0, T]$. A similar conclusion holds for a certain class of nonlinear hyperbolic equations. Some remarks are also made concerning the singular solutions of $\square u = \lambda u^k$, which arises in the theory of self-coupled photon fields, especially with respect to the nonanalyticity of their dependence on λ at $\lambda = 0$. (Received March 12, 1954.)

422. Walter Rudin: *Multiplicative groups of analytic functions.*

For a proper subdomain D of the Riemann sphere, $G(D)$ denotes the multiplicative group of all nonvanishing regular single-valued analytic functions f on D, normalized by $f(z_0) = 1$ for some fixed $z_0 \in D$. $H(D)$ is the subgroup of $G(D)$ which consists of those f for which the equation $g^n = f$ has a solution $g \in G(D)$, for every integer $n \neq 0$; $f \in H(D)$ if and only if f has a single-valued logarithm in D. Results: (1) For any two domains D_1 and D_2, $H(D_1)$ and $H(D_2)$ are isomorphic. (2) $G(D)$ is the direct product of H and $G(D)/H$. (3) If the complement of D has $k+1$ components, $G(D)/H$ is the direct product of k infinite cyclic groups. (4) If the complement of D has infinitely many (countable or not) components, $G(D)/H$ is isomorphic to the additive group of all integer-valued functions on a countable space. It follows that the algebraic structure of $G(D)$ determines, and is determined by, the connectivity of D. (Received March 9, 1954.)

423. J. B. Serrin: *A uniqueness theorem for the parabolic equation $u_t = a(x)u_{xx} + b(x)u_x + c(x)u$.* Preliminary report.

Suppose that the coefficients of the equation $(*)$ $u_t = a(x)u_{xx} + b(x)u_x + c(x)u$ are bounded and Hölder continuous for $-\infty < x < +\infty$, and that $a(x) \geq k$ where k is a positive constant. Let S denote the strip $-\infty < x < +\infty$, $0 < t < T$ in the (xt)-plane and $\overline{S}$ its closure (i.e. the strip $0 \leq t \leq T$). Then if $u(x, t)$ is a non-negative solution of $(*)$ in S which is continuous in $\overline{S}$, it can be represented in terms of its initial values $u(x, 0) = f(x)$ by a functional operator $P(f)$. It follows that such a solution $u(x, t)$ is uniquely prescribed by its initial values. This represents a partial extension of a theorem of Widder concerning non-negative solutions of the equation of heat conduction (Trans. Amer. Math. Soc. vol. 55 (1944)). (Received March 10, 1954.)

424t. V. L. Shapiro: *Localization of conjugate multiple trigonometric series.*

Let $T = \sum_m a_m e^{i(m,x)}$ be an n-dimensional trigonometric series where $(m, x) = m_1 x_1 + \cdots + m_n x_n$, and let $K(x) = P_g(x)/|x|^{-(n+g)}$ where $P_g(x)$ is a homogeneous polynomial of degree g which satisfies Laplace's equation $\Delta P = 0$. Then Calderón and Zygmund have shown that to the trigonometric series T one can associate a series $\overline{T}$, called the conjugate series of T with respect to the spherical harmonic kernel $K(x)$, which is an extension of the classical one-dimensional case. $\overline{T}$ is defined by $\overline{T} = \sum_m a_m \widehat{K}(m) e^{i(m,x)}$ where $\widehat{K}(u)$ is the principal-valued Fourier transform of $K(x)$, recently evaluated by Bochner. Using the technique of formal multiplication of series developed by Rajchman and Zygmund, and also Berkovitz, this paper studies the localization properties of the above conjugate series. In particular if a_m, the coefficients of the T, are $o(|m|^p)$, p an integer not less than $-(n-1)$, a function $F(x) = a_0|x|^{2q}/K_q + (-1)^q \sum_{m \neq 0} a_m |m|^{-q} e^{i(m,x)}$ can be associated to T where q is the integer $[(p+n)/2] + 1$ and $\Delta^q |x|^{2q} = K_q$. It is then shown in this paper that if $F(x)$ is in class $C^{(2q+n+2)}$ on a domain D contained in the fundamental cube R, then $\overline{T}$ is uniformly spherically summable $(C, p+n-1)$ in every interior closed subdomain of D. (Received March 9, 1954.)

425. F. M. Stewart: *Periodic solutions of a nonlinear wave equation.*

The differential equation (A) $u_{tt} - u_{xx} = \mu(u^3 + f(x)) \sin t)$ is replaced by an ordinary differential equation, (B) $Z' = \mu F(Z, t)$, in a space of sequences. Following Fatou [Bull. Soc. Math. France vol. 56 (1928) pp. 98–139] one can find a Z_0 such that $\overline{F}(Z_0) = (1/2\pi)\int_0^{2\pi} F(Z_0, t)dt = 0$. It is proved that if μ is sufficiently small, then (B) has a periodic solution near Z_0. Difficulties, caused by the fact that the linear part of $\overline{F}$ is singular, are overcome by using several norms in the space of sequences. Using the periodic solution of (B) it is easy to construct a periodic solution of (A). (Received March 9, 1954.)

426. C. T. Taam: *Criteria of boundedness of the solutions of nonlinear differential equations.*

Let r and p_i be real-valued continuous functions of x on $I: a \leq x < \infty$, r being positive. Let also rp_i be absolutely continuous on I. Consider only the real-valued solutions of (A): $(ry')' + \sum_{i=1}^n p_i y^{2i-1} = 0$. Denote by f'_+ the max $(f', 0)$ and by f'_- the min $(f', 0)$. It is proved that if (1) $rp_i > 0$ on I, $i = 1, 2, \cdots, n$, (2) $(rp_i)'_-/rp_i$ belongs to $L(a, \infty)$ for some $i = k$ and $(rp_i)'_+/rp_i$ belongs to $L(a, \infty)$ for all $i \neq k$, then

every solution of (A) is bounded on I. Other results derived from this theorem are also given. Let (B) denote what results in (A) if the power of y in (A) is i and n is a positive odd integer. The following theorem is also obtained: Every solution y of (B) and ry' are bounded on I if $(rp_i)'$ belongs to $L(a, \infty)$ for $i = 1, 2, \cdots, n$ and rp_n has a positive lower bound on I. (Received March 10, 1954.)

427. J. L. Walsh and D. M. Young (p): *Lipschitz conditions for harmonic and discrete harmonic functions.*

Let R denote a plane region whose boundary S is a Jordan curve, let $u(z)$ be harmonic in R, continuous in $\Omega = R + S$, and let $u(z) \in \text{Lip } \alpha$ on S, $0 < \alpha < 1$, in the sense $|u(z_1) - u(z_2)| \leq L |z_1 - z_2|^\alpha$ (L constant), for $z_1, z_2 \in S$. If R is a disk or half plane, then $u(z) \in \text{Lip } \alpha$ in Ω. This conclusion is shown to hold when R is a semi-infinite strip or a rectangle. An "accordion folding process" is used, which studies, for instance, the functions $u(x, y) - u(2-x, y) + u(2+x, y) - u(4-x, y) + u(4+x, y) - \cdots$ in the strip $0 \leq x \leq 1$, $y \geq 0$, if $u(x, y)$ is defined for $x \geq 0$, $y \geq 0$. Next, let $U(z)$ be discrete harmonic in R_L and assume the same values as $u(z)$ on S_L. (Here R_L and S_L denote respectively the set of points (mh, nh) (m, n integers) belonging to R and S.) An inequality of Allen and Murdoch (Proc. Amer. Math. Soc. (1953)) is used to show $U(z) \in \text{Lip } \alpha$ uniformly with respect to h in $\Omega_L = R_L + S_L$ if R is Im $(z) > 0$. The conclusion also holds when R is a semi-infinite strip or a rectangle. For the rectangle the difference $|U(z) - u(z)|$ $= O(h^{2\alpha/(\alpha+3)})$ uniformly in Ω_L as $h \to 0$. This improves a previous result of the authors. (Received March 9, 1954.)

428. František Wolf: *Simplicity of spectra in general operators.*

Let $E(M)$ be a spectral measure, i.e. a mapping of Borel sets in the plane into uniformly bounded idempotent operators in Hilbert space, completely additive with the usual other conditions. Then, if there exists a generator g such that $E(M)g$, considered for all Borel sets, generates all of Y, then $E(M)$ will be called simple. Then $A = \int \lambda dE(\lambda)$ is a scalar operator in Dunford's sense (cf. Pacific Journal of Mathematics vol. 2 (1952) p. 559) with a simple spectrum: Any bounded B commutative with A is of the form $\int b(\lambda) dE(\lambda)$, i.e. a function of A, $b(A)$. Any spectral operator (cf. Dunford loc. cit.) admitting $E(M)$ as its spectral measure is a scalar operator. By Wermer's result (to appear in the Pacific Journal of Mathematics (1954)) there is no loss of generality to suppose that $E(M)$ is a mapping into projections. Then the condition stated guarantees that there exists (cf. Akhiezer-Glazman, *Theory of lin. operators*, in Russian, p. 269) an isometric transformation of our space into a $L^2(d\sigma)$ and that A transforms into $x \cdot$, the multiplicative operator. Any operator commutative with A, by a well known theorem (cf., i.e., Akhiezer-Glazman, loc. cit. p. 300) must be a function of A, and so a scalar operator. (Received March 10, 1954.)

APPLIED MATHEMATICS

429. Milton Abramowitz and H. A. Antosiewicz (p): *Coulomb wave functions in the transition region.*

The regular and irregular solutions, $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$, of the Coulomb wave equation $y'' + (1 - 2\eta \rho^{-1} - L(L+1)\rho^{-2})y = 0$ are obtained for $L = 0$ and values of η and ρ in terms of the variable $t = (2\eta - \rho)(2\eta)^{-1/3}$ under the assumption that $|t|$ is small. These solutions are expressed in the form $\theta(t, \eta) U(t) + \psi(t, \eta) U'(t)$ where $U(t)$ is a solution of the equation $U''' - tU = 0$ and $\theta(t, \eta)$, $\psi(t, \eta)$ are power series in t so determined that these solutions yield for $t = 0$ the values of F_0, G_0 at $\rho = 2\eta$, respectively. By

identifying $U(t)$ with the Airy integrals $Ai(t)$ for $F_0(\eta, \rho)$ and $Bi(t)$ for $G_0(\eta, \rho)$, the power series obtained for $\theta(t, \eta)$, $\psi(t, \eta)$ are identical in both cases and converge rapidly for sufficiently small $|t|$. The values of $F_L(\eta, \rho)$, $G_L(\eta, \rho)$ with $L>0$ may be obtained from those of $F_0(\eta, \rho)$, $G_0(\eta, \rho)$ by use of recurrence relations satisfied by $F_L(\eta, \rho)$, $G_L(\eta, \rho)$ without loss of accuracy for the pertinent range of values of η and ρ. This method may be generalized to equations $y''+\phi(t, \lambda)y=0$ where $\phi(t, \lambda)\to t^k$, $k\geqq0$, as $\lambda\to\infty$ and $\phi(t, \lambda)$ is analytic in t at $t=0$. (Received March 8, 1954.)

430t. F. H. Brownell: *Perturbation of the n-dimensional Schrödinger differential operator.*

Consider the Schrödinger operator $[Au](x) = -\nabla^2u(x)+V(x)u(x)$ for $u\in D$ dense in $L_2(R_n)$ over $x\in R_n$, n-dimensional euclidean space, where ∇^2 is the Laplacian $\sum_{j=1}^{n}\partial^2/\partial x_j^2$ and $V(x)$ is an arbitrary real-valued, measurable function over R_n. For $n=3$, if $\{$ess $\sup_{x\in R_3}\int_{R_3}(|V(y)|/4\pi\||x-y\|)d\mu_3(y)\}<1$, then a natural self-adjoint extension of A has the same spectrum as that of the unperturbed operator A_0 defined by setting $V(x)\equiv0$, namely the point spectrum is void and the continuous spectrum is precisely the positive axis $[0, +\infty)$. Similar results hold for $n>3$, but the conditions on $V(x)$ become more complicated. The question of unitary equivalence of the extensions of A and A_0 is also considered. (Received March 12, 1954.)

431. J. B. Diaz (p) and G. S. S. Ludford: *A transonic approximation.*

This approximation is based on obtaining that pressure-density relation which best fits that of a polytropic gas near the sonic point, whilst retaining the analytical simplicity of the Tricomi gas. Third order contact in the pressure-density relation can be obtained at the sonic point, and there still remains a parameter, which can be chosen in two ways so as to give good supersonic agreement. These two choices give practically identical results. For this approximation the general solution of the partial differential equation for the stream-function is expressible in terms of Bessel functions, as also are the product solutions. (Received March 10, 1954.)

432. L. C. Maximon: *Indefinite integrals involving the special functions.*

In the solution of one-dimensional linear homogeneous time dependent partial differential equations one may take the transform with respect to time and solve the resulting ordinary inhomogeneous differential equation, obtaining the transform in terms of indefinite integrals involving solutions of the homogeneous transformed equation. If the boundary and initial conditions are such that the solution to the original problem is actually time independent, its time-transform is then trivial and one may reverse the procedure just mentioned to solve for the indefinite integrals. Following this procedure a general method of evaluating in closed form a large class of integrals involving the special functions of mathematical physics has been developed and a number of typical examples are given. The method is also extended to evaluate integrals containing functions which are solutions of nth order linear differential equations. (Received March 10, 1954.)

433. Cathleen S. Morawetz: *A nonexistence theorem for transonic flows past a profile.* Preliminary report.

Let there be given a sufficiently smooth, two-dimensional, steady, irrotational

compressible flow F_0, with potential ϕ_0 and complex velocity $qe^{-i\theta}$ about a convex profile P_0. The free stream Mach number of the flow is less than 1 but there is a finite supersonic region. The profile and flow are symmetric about the x and y axes. Let S be the segment of the profile cut out by the two Mach lines issuing from the intersection of the sonic curve with the y-axis. Let ϕ_1 be the velocity potential of an irrotational, steady, symmetric, continuous flow, with the same free stream Mach number as F_0, around a profile P_1 coinciding with P_0 except on S and very close to P_0. If products of the first two derivatives of $\omega = \phi_1 - \phi_0$ are neglected, ω satisfies the equation (1) $K(\sigma)\omega_{\theta\theta} + \omega_{\sigma\sigma} = 0$ with $K(\sigma) \gtrless 0$ for $q \gtrless c$, $\sigma = \sigma(q)$, and the boundary condition (2) $K(\sigma)\omega_\theta d\sigma - \omega_\sigma d\theta = 0$ on the images of $P_0 - S$ and $y = 0$ in the (θ, σ)-plane. We prove that there exists only one solution ω satisfying (1) and (2). Hence there exists at most a one-parameter family of profiles P_1 for which a continuous flow exists. This is a rigorous proof of the conjecture, frequently stated in the literature, that the perturbation problem belonging to a transonic flow has no solution in general. (Received March 23, 1954.)

434. Charles Saltzer: *Two-terminal relay and switching circuits.* Preliminary report.

A topological method is given for determining the Boolean hindrance function for a two-terminal relay and switching circuit from the incidence matrix of the graph of the circuit. (Received March 10, 1954.)

435. Raymond Sedney: *A maximum principle for the drag in linearized airfoil theory.* Preliminary report.

A function space is established for suitably restricted functions representing the pressure and angle of attack distributions on a lifting surface. Using linearized airfoil theory, the scalar product is defined as one-half the mutual interference drag of two distributions; the metric is then the square-root of the drag of a distribution. This idea was first introduced by E. W. Graham and P. A. Lagerstrom. Using simple geometrical reasoning, a maximum principle involving the drag is found. This can be used to obtain approximate solutions to the lifting surface integral equation, similar to the Rayleigh-Ritz procedure. A workable procedure has been found only for the case of subsonic flow. When used as an iteration method, the procedure gives an increasing sequence of lower bounds for the drag. However, singularities in the pressure distribution must be excluded so that the drag does not include leading edge suction. (Received March 10, 1954.)

436t. R. L. Sternberg: *Elementary methods in the numerical design of microwave dielectric lenses.* Preliminary report.

A number of apparently new techniques are presented which facilitate the design of zoned or unzoned axially symmetric dielectric lenses for microwave antenna applications in which it is required to produce an off axis beam from an off axis focus when the refractive index, the wave length, the beam angle, the focal distance and the lens diameter are preassigned. Among other results obtained in the meridional plane it is shown: (i) that the focus must be located on a particular hyperbola, (ii) that the locations of the zone extremities are unique and vary with the wave length, the beam angle, the focal distance, and the lens diameter, and (iii) that the required nonspherical surfaces may be fitted in a nontrivial manner to control several meridional rays perfectly including, in particular, the meridional rays through the zone and lens

extremities. Elementary methods are indicated suitable for carrying out the required calculations. Applications to scanning antenna problems are briefly discussed. (Received March 5, 1954.)

437t. **R. L. Sternberg, J. S. Shipman, and Hyman Kaufman:** *Tables of Bennett functions for the two-frequency modulation product problem for the half-wave square law rectifier.* Preliminary report.

Eight decimal tables of the functions $A_{mn}^{(2)}(k) = (2/\pi^2) \iint_R (\cos u + k \cos v)^2 \cos mu du$ $\cdot \cos nv dv$, R: $\cos u + k \cos v \geq 0$, $0 \leq u$, $v \leq \pi$, for $k = 0.02$ (0.02) 1.0 and $m + n \leq 5$ are provided for application to the problem referred to in the title (see W. R. Bennett, *New results in the calculation of modulation products*, Bell System Technical Journal vol. 12 (1933) pp. 228–243). The evaluation of the functions $A_{mn}^{(2)}(k)$ is carried out by means of power series expansions and finite formulas in complete elliptic integrals while the actual arithmetic is performed on I.B.M. equipment. New recursion identities by means of which the higher order functions $A_{mn}^{(2)}(k)$ can be expressed in terms of those tabulated are given. Bennett functions $A_{mn}^{\nu}(k)$ of the νth kind are defined and recursion identities between the $A_{mn}^{(\nu+1)}(k)$ and $A_{m'n'}^{(\nu)}(k)$ are derived by means of which a number of the higher kind functions may be evaluated readily in terms of the second kind functions tabulated. (Received February 22, 1954.)

438t. **R. L. Sternberg, J. S. Shipman, and W. B. Thurston:** *Tables of Bennett functions for the two-frequency modulation product problem for the half-wave linear rectifier.*

Eight decimal tables of the functions $A_{mn}(k) = (2/\pi^2) \iint_R (\cos u + k \cos v) \cos mu du$ $\cdot \cos nv dv$, R: $\cos u + k \cos v \geq 0$, $0 \leq u$, $v \leq \pi$, for $k = 0.02$ (0.02) 1.0 and $m + n \leq 4$ are provided for application to the problem referred to in the title (see W. R. Bennett, *New results in the calculation of modulation products*, Bell System Technical Journal vol. 12 (1933) pp. 228–243). The evaluation of the functions $A_{mn}(k)$ is carried out by means of power series expansions and finite formulas in complete elliptic integrals while the actual arithmetic is performed on I.B.M. equipment. Known recursion identities by means of which the higher order functions $A_{mn}(k)$ can be expressed in terms of those tabulated are noted. (Received February 22, 1954.)

439. **H. F. Weinberger:** *A Rayleigh-Ritz procedure with arbitrary error estimate for differential operators with finite perturbations.*

Let B be a positive, symmetric operator with discrete spectrum, having known eigenvectors u_i and eigenvalues $\mu_i \to \infty$. Let A be an operator whose eigenvalues λ_i are to be approximated, and which satisfies $B \leq A \leq B + cI$, where c is a known constant and I the identity operator. Let $\lambda_i^{(N)}$ be the ith eigenvalue obtained by applying the Rayleigh-Ritz method to A, using the first N eigenvectors of B as "coordinate vectors." Then it is well known that $\lambda_i \leq \lambda_i^{(N)}$. The following inequality then serves as an error estimate for the approximation of $\lambda_i^{(N)}$ to $\lambda_i : \lambda_i \geq \lambda_i^{(N)} - c^2 [4\mu_{N+1} - 4\lambda_i^{(N)} + 2c]^{-1}$. Since $\lambda_i^{(N)}$ decreases with N^{-1}, it may be replaced in the error term by an earlier Rayleigh-Ritz bound, the simplest being $\lambda_i^{(0)}$. Then the error term may be made arbitrarily small by choosing N sufficiently large, since $\mu_{N+1} \to \infty$. The above inequality is obtained by first proving a stricter but more complicated inequality for the special case where A and B are 2×2 matrices. This inequality is then extended to operators in Hilbert space by decomposing vectors into their projections into the subspaces $[u_1, \cdots, u_N]$ and its complement. (For a similar procedure, see the proof of Aron-

szajn's Inequality in Hamburger and Grimshaw, *Linear transformations in n-dimensional vector space*, Cambridge, 1951, p. 76.) The inequality so obtained is then simplified at the price of some precision. The method here presented is applicable to a Schroedinger operator differing by a bounded potential function from a known Schroedinger operator with discrete spectrum. (Received March 11, 1954.)

GEOMETRY

440. H. S. M. Coxeter: *Arrangements of equal spheres in non-Euclidean spaces.*

Fejes Tóth (*Lagerungen in der Ebene, auf der Kugel und im Raum*, Berlin, 1953, p. 114), in his work on arrangements of equal circles, proved that the elliptic plane cannot be packed as densely, nor covered as thinly, as the Euclidean plane. Accordingly, it is interesting to find a different state of affairs among the arrangements of equal spheres in space. It is found that, in elliptic space, sixty spheres of radius $\pi/10$, each touching twelve others, form a packing of density $0.774 \cdots$, and that somewhat larger spheres having the same centers form a covering of density $1.439 \cdots$, whereas in Euclidean space the maximum packing-density and the minimum covering-density are almost certainly $0.740 \cdots$ and $1.464 \cdots$ (Fejes Tóth, op. cit. pp. 171, 174). On the other hand, greater packing-densities and smaller covering-densities occur in the hyperbolic plane (Fejes Tóth, op. cit. p. 157) and in hyperbolic space. In fact, one can find a packing of horospheres (spheres of infinite radius) and a covering by horospheres, of respective densities $(1+2^{-2}-4^{-2}-5^{-2}+7^{-2}+8^{-2} - \cdots)^{-1}=0.855 \cdots$, $(1-2^{-2}+4^{-2}-5^{-2}+7^{-2}-8^{-2}+ \cdots)^{-1}=1.282 \cdots$. These values were obtained with the aid of the following lemma: The volume of any solid "sector" of a horosphere (in natural measure) is equal to one-half the area of its horospherical boundary. (Received March 2, 1954.)

441. C. C. Hsiung: *Some integral formulas for closed hypersurfaces.*

Let V^n be a hypersurface with a closed boundary V^{n-1} twice differentiably imbedded in a Euclidean space E^{n+1} of $n+1$ $(n \geq 2)$ dimensions, and let M_α be the αth mean curvature of V^n at a point P defined by the elementary symmetric function $M_\alpha = \sum \kappa_1 \kappa_2 \cdots \kappa_\alpha$ $(\alpha = 1, \cdots, n)$, where $\kappa_1, \kappa_2, \cdots, \kappa_n$ are the n principal curvatures and M_n the Gaussian curvature K of V^n at P. The purpose of this paper is to derive for V^n two integral formulas from which follow immediately for a closed orientable V^n the following two formulas, which were obtained by H. Minkowski for a closed convex hypersurface V^n with $n=2$: $nA + \int_{V^n} M_1 p dA = 0$, $n \int_{V^n} K p dA + \int_{V^n} M_{n-1} dA = 0$, where dA is the area element of V^n at P and p an oriented distance from a fixed point 0 to the tangent hyperplane of V^n at P. From the above two formulas some simple conditions for a closed orientable hypersurface V^n to be a hypersphere are deduced. (Received March 8, 1954.)

442. C. C. Hsiung and C. B. Sensenig (p): *On convex surfaces.*

Let S_1 (S_2) be a convex surface of class ≥ 3 with a plane boundary C_1 (C_2) in an ordinary Euclidean space. If S_1 and S_2 satisfy a certain additional boundary condition, a one-to-one correspondence ϕ between the points of S_1 and S_2 is defined under which any two corresponding points P_1, P_2 of S_1, S_2 have the same parameters u and v. Let B_1, B_2 be the plane areas bounded by C_1, C_2 respectively, and let $K_1(u, v)$, $K_2(u, v)$ and dA_1, dA_2 be respectively the Gaussian curvatures and the area elements of S_1, S_2 at P_1, P_2. It is shown that if S_1, S_2 are in a correspondence ϕ and $dA_1 = dA_2$, $K_1 \geq K_2$

for all corresponding points of S_1, S_2, then $B_1 \leqq B_2$. Moreover, $B_1 = B_2$ implies that S_1 and S_2 are congruent if S_1 and S_2 are closed and analytic or are surfaces of revolution with axes of revolution perpendicular to the planes of C_1 and C_2 respectively. The first part of these results is an analogue of a theorem of A. Schur on two curves. (Received March 8, 1954.)

443. Valdemars Punga: *Theorem on the construction of affine connection in Cartan space of line elements.*

Cartan space of line elements (E. Cartan, *Les espaces de Finsler* II) is a set of objects in (1-1) correspondence with $2n$ ordered real numbers (x^1, x^2, $\cdots$, x^n; p^1, $\cdots$, p^n) or simply (x^α, p^α) or (x, p), where line elements (x, p) and (x, cp), $c > 0$, are considered equal. p^α transforms as contravariant vector. The covariant differential (δ) of a vector $V^\alpha(x, p)$ is given by $\delta V^\alpha = (\nabla_\gamma^1 V^\alpha) dx^\gamma + (\nabla_\gamma^2 V^\alpha) \delta p^\gamma = dv^\alpha + \Gamma_{\beta\gamma}^\alpha v^\gamma dx^\beta + C_{\beta\gamma}^\alpha v^\gamma \delta p^\beta$, where $\Gamma_{\beta\gamma}^\alpha(x, p)$ is homogeneous of degree 0 in p and tensor $C_{\beta\gamma}^\alpha(x, p)$ is homogeneous of degree (-1) in p. *Theorem*: Any five tensors $S_{\beta\gamma}^\alpha(x, p)$, $\overline{S}_{\beta\gamma}^\alpha(x, p)$, $Q_{\gamma\alpha\beta}(x, p)$, $\overline{Q}_{\gamma\alpha\beta}(x, p)$ and $f_{\alpha\beta}(x, p)$, where (a) all five tensors are homogeneous functions of degree 0 in p, (b) $S_{\beta\gamma}^\alpha$ and $\overline{S}_{\beta\gamma}^\alpha$ are skew-symmetric in β and γ indices, (c) $Q_{\gamma\alpha\beta}$ and $\overline{Q}_{\gamma\alpha\beta}$ are symmetric in α and β indices, (d) $f_{\alpha\beta} = \partial^2 f / \partial p^\alpha \partial p^\beta$ with determinant $|f_{\alpha\beta}| \neq 0$, determine uniquely an affine connection ($\Gamma_{\beta\gamma}^\alpha$, $C_{\beta\gamma}^\alpha$) which is such that $\nabla_\gamma^1 f_{\alpha\beta} = Q_{\gamma\alpha\beta}$, $\nabla_\gamma^2 f_{\alpha\beta} = \overline{Q}_{\gamma\alpha\beta}$, ($\Gamma_{\beta\gamma}^\alpha - \Gamma_{\gamma\beta}^\alpha$)$/2 = S_{\beta\gamma}^\alpha$, ($C_{\beta\gamma}^\alpha - C_{\gamma\beta}^\alpha$)$/2 = \overline{S}_{\beta\gamma}^\alpha$. The author proved the theorem by expressing $\Gamma_{\beta\gamma}^\alpha$ and $C_{\beta\gamma}^\alpha$ in terms of given tensors. (Received February 24, 1954.)

LOGIC AND FOUNDATIONS

444t. K. J. J. Hintikka: *A reduction in the simple theory of types.*

To each closed formula K of the simple theory of (finite) types one can associate, by an effective rule, a formula $\phi(K)$ of the first-order functional calculus in such a way that K is valid (in the sense of a standard truth-definition) if and only if the formula ($D \supset \phi(K)$) is valid, and K is logically false if and only if ($D \supset \sim \phi(K)$) is valid. Here, D is a constant formula of the second-order functional calculus containing only one quantified monadic predicate (functional) variable (in addition to a few free predicate variables and to a number of bound individual variables). If one allows for nonstandard models, then D may be reduced further so as to contain no bound predicate variables. (Received March 19, 1954.)

445t. J. R. Shoenfield: *Models of formal systems.*

Let P be a system of axioms for set theory, and let P' be the system formed from P by adding an additional axiom A. Suppose that the statement "P in consistent" is provable in P'. By a theorem of Gödel, if P is consistent, A is not provable in P. Moreover, if P is ω-consistent, then the statement "If P is consistent, then P' is consistent" is not provable in P. This shows that A is independent of P, and that the consistency of A with P is not provable by methods available in P. The above may be applied to give new proofs of certain independence results, and shows at the same time why corresponding consistency results are unavailable. For example, P may be taken as Zermelo-Fraenkel set theory, and A may be taken as the axiom asserting the existence of a strongly inaccessible cardinal. The main step, in each case, consists in developing the theory of models in P' so that "P is consistent" may be proved in P'. (Received March 12, 1954.)

STATISTICS AND PROBABILITY

446. M. D. Kruskal: *The expected number of components under a random mapping function.*

Let f be a mapping of a finite set S of N elements into itself. A subset T is *invariant* if $f(T) \subset T$ and $f^{-1}(T) \subset T$. A *component* is a minimal non-null invariant subset. (The components form a partitioning of S.) Metropolis and Ulam (Amer. Math. Monthly vol. 60 (1953) p. 252) have raised the problem of finding the expected number E of components if f is a uniformly distributed random function (i.e. if f has probability N^{-N} of being any specific mapping function). The author proves that $E = \sum_{n=1}^{N} N!/(N-n)!nN^n$ and that $E - (\log 2N + C)/2 \to 0$ as $N \to \infty$, where C is Euler's constant. (Received March 8, 1954.)

TOPOLOGY

447t. A. F. Batholomay: *The Serre group $E_2^{p,q}$.*

The terms E_r of the "spectral sequence" [*Homologie singulière des espaces fibrés*, J.-P. Serre, Ann. of Math. (1951)] associated with the filtration $\cdots \subset A^p \subset A^{p+1} \subset \cdots$ of a differential group (A, d) are such that $E_{r+1} = H(E_r)$, where H denotes the homology group with respect to the operators (d_r) defined by d. If group A is graded, then each E_r is bi-graded; i.e., $E_r = \sum_{p,q} E_r^{p,q}$. The important term $E_2^{p,q}$ is expressible as $H(H_{p+q}(A^p/A^{p-1}))$. Another form for $E_2^{p,q}$ is known to be $\bar{E}_2^{p,q} = $ image f, where f: $H_{p+q}(A^p/A^{p-2}) \to H_{p+q}(A^{p+1}/A^{p-1})$. An explicit form is obtained for such a homomorphism, at the same time that a proof by construction of the equivalence of the two forms is accomplished. First, the mapping $f_2 = (i \circ j')$ is defined: $A^p/A^{p-2} \to^{i'} A^p/A^{p-1} \to^i A^{p+1}/A^{p-1}$, where the induced homomorphisms i_*, j'_* are identified with the homology sequences of (A^{p+1}, A^p, A^{p-1}) and (A^p, A^{p-1}, A^{p-2}), respectively. After identifying d^p and d_1^{p+1} with corresponding homomorphisms in the homology sequences, the equivalence of the two forms then follows by observing that $E^{p,q}$ = (kernel d_1^p/image d^{p+1}) = (image j'_*/kernel i^*). (Received March 3, 1954.)

448t. Morton Brown: *N homogeneity implies $N-1$ homogeneity.*

A topological space S is N homogeneous if for any N distinct points $p_1, p_2, \cdots, p_N$ and any N distinct points $q_1, q_2, \cdots, q_N$ there exists a homeomorphism of S onto itself taking $p_1, p_2, \cdots, p_N$ onto some permutation of $q_1, q_2, \cdots, q_N$. It is a natural question to ask whether N homogeneity implies $N-1$ homogeneity. The author proves the more general theorem that if T is a group of 1-1 transformations of an infinite set onto itself such that for any two subsets X, Y, each containing N distinct points, there exists a $t \in T$ such that $t(X) = Y$, then for any two sets X', Y', each containing $N-1$ distinct points, there exists a $t' \in T$ such that $t'(X') = Y'$. (Received March 10, 1954.)

449. Eldon Dyer: *Continuous collections of decomposable continua on a spherical surface.*

Let S denote a spherical surface and G denote a continuous collection of mutually exclusive decomposable continua on S which is a continuum with respect to its elements. Then G with respect to its elements is a continuum such that each of its non-degenerate subcontinua contains uncountably many of its locally separating points. (O. G. Harrold, Jr. presented a list of characterizations of such continua in *Lectures in*

topology, University of Michigan Conference of 1940, p. 248). If G fills up S, then with respect to its elements it is a dendron. For each dendron D there is a continuous collection of mutually exclusive decomposable continua filling up S which is with respect to its elements topologically equivalent to D. If G fills up S and none of its elements is the boundary of three of its complementary domains, G is an arc with respect to its elements. R. D. Anderson has obtained results of a similar nature for continuous collections of nondegenerate continuous curves in the plane (Proc. Amer. Math. Soc. vol. 3 (1952) pp. 647–657). (Received March 10, 1954.)

450. E. R. Fadell: *A property of compact absolute neighborhood retracts.*

A continuous mapping $f: X \to X$ is said to be a ϕ-mapping if for every $x \in X$, there exists an $x' \in X$ different from x such that $f(x') \neq x'$ and x' does not separate x and $f(x')$. For example, every fixed point free (fpf) mapping $f: X \to X$, where X is a continuum (metric), is a ϕ-mapping and *any* mapping $g: X \to X$, $g \neq 1$, where X is a cyclic (no cut points) continuum, is also a ϕ-mapping. The main result of this paper is the following theorem. Let $f: X \to X$ denote a ϕ-mapping, where X is a compact ANR (metric, not necessarily connected). Then X admits of a mapping $g: X \to X$ such that $g(X) = X$ and $g \sim f$ with homotopy H such that the set of fixed points under H_t, $0 \leq t \leq 1$, is the set of fixed points of f. In particular, therefore, if X is a compact, connected ANR and f is fpf, then f is continuously deformable into a fpf mapping g which is onto. Also, if $f: X \to X$ is *any* mapping from a cyclic, connected, compact ANR, then f is continuously deformable into a mapping g which is onto and possesses the same set of fixed points as f. (Received February 26, 1954.)

451*t*. Gertrude I. Heller: *Interrelations between directed sets and filters.*

Moore and Smith in their general theory of limits (both authors, Amer. J. Math. (1952); the latter, National Mathematics Magazine (1938)) used the notion of directed sets. Frechet based his limit theory on the notion of filter, later elaborated by Bourbaki (*Topologie générale*, Paris, 1940). Since Marlow (Bull. Amer. Math. Soc. vol. 58) used directed sets to define upper and lower limits, the two theories became so parallel that the question of their interrelationships imposed itself. The paper compares the two theories and the structures on which they are based. The main result is: While any directed set obviously generates a filter, namely the collection of residuals (and therefore any theory of limits based on filters will necessarily contain the corresponding theory based on directed sets), the converse is not true. A necessary and sufficient condition for a filter F on a set E to be obtainable as the collection of residuals of some directing ordering of E is that F have a basis B with the property that the intersection of any collection of sets of B is either empty or belongs to F. An example of a filter which does not fulfill this condition was furnished by W. L. Chow. (Received March 10, 1954.)

452*t*. L. F. McAuley: *A relation between perfect separability, completeness, and normality in semi-metric spaces.*

Let S denote a regular semi-metric topological space. C. W. Vickery has stated (Bull. Amer. Math. Soc. vol. 46 (1940) p. 433) that S is a Moore space. In this paper, it is proved that S is not necessarily a Moore space. Indeed, S may have many properties such as complete separability (very subset of S is separable), complete normal-

ity, and weak completeness but S fails to be either a Moore space or a metric space. While it is shown that a complete Moore space is not necessarily a strongly complete space S, it is proved that if S is strongly complete and completely separable, then S is a perfectly separable complete Moore space. Other theorems are proved which include affirmative answers to questions raised by L. W. Cohen (Duke Math. J. vol. 5 (1939) p. 183) and W. A. Wilson (Amer. J. Math. vol. 53 (1931) p. 366). (Received March 8, 1954.)

453. Mary E. Rudin: *A connected set such that the complement of every connected subset is countable.*

Paul Erdös (*Some remarks on connected sets*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 442–446), raised the question of the existence of a nondegenerate connected set such that the complement of every nondegenerate connected subset is countable. This paper shows that, if the hypothesis of the continuum is true, there exists such a set in the plane. (Received March 9, 1954.)

454t. Hans Samelson: *Groups and spaces of loops.*

A construction is given which shows that every topological group with a universal bundle (contractible principal bundle) is homotopically equivalent to the space of loops of the corresponding classifying space (base space of the bundle); the equivalence includes the operations of multiplication and inversion. From this and from known properties of homotopy groups of spheres it is shown that the commutator map of the group of quaternions of norm 1 $((x, y) \rightarrow xyx^{-1}y^{-1})$ is not null-homotopic, answering a question of Eilenberg's. (Another proof was found independently by G. W. Whitehead.) Another application: The generator z of the nth homology group of the Eilenberg-MacLane space $K(Z, n)$ satisfies, for odd n, $z * z = 0$, where $*$ is the Pontryagin product. This work was sponsored by the National Science Foundation. (Received January 21, 1954.)

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