THE JUNE MEETING IN MISSOULA

The four hundred ninety-third meeting of the American Mathematical Society was held at Montana State University, Missoula, Montana, on June 20, 1953. There were approximately seventy registrations, including the following forty-one members of the Society:

T. M. Apostol, B. H. Arnold, M. G. Arsove, Wilfred E. Barnes, R. A. Beaumont, L. G. Butler, Harold Chatland, P. A. Clement, C. M. Cramlet, R. Y. Dean, D. B. Dekker, F. E. Ehlers, Paul Erdös, R. M. Gordon, S. G. Hacker, M. E. Haller, C. A. Hayes, R. D. James, T. R. Jenkins, H. E. Kinerk, J. M. Kingston, M. S. Knebelman, R. B. Leipnik, A. T. Lonseth, A. S. Merrill, W. E. Milne, Leo Moser, W. M. Myers, Jr., Ivan Niven, A. R. Noble, I. L. Olson, T. G. Ostrom, W. T. Putney, R. A. Rosenbaum, R. G. Selfridge, A. J. Smith, W. M. Stone, J. R. Vatnsdal, R. M. Winger, R. J. Wisner, F. H. Young.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor A. T. Lonseth of Oregon State College addressed the Society on *Approximate solutions of Fredholm-type integral equations*. The speaker was introduced by Professor R. M. Winger. The session for contributed papers began at ten o'clock Saturday morning. The presiding officer was Professor M. S. Knebelman. On June 19 there was the Pacific Northwest Sectional Meeting of the Mathematical Association of America. The hour speaker was Professor Leo Moser who addressed the Association on *The distribution of quadratic residues*. There was a joint dinner for members of the Society and the Association and their guests at the Hotel Florence, followed by a social hour at the Faculty Club.

Abstracts of the papers presented follow. Those whose abstract numbers are followed by "*t*" were presented by title. Paper number 530 was presented by Dr. J. E. Maxfield, number 534 by Professor Erdös, number 536 by Professor Hayes, and number 539 by Professor Leipnik. Professor Ohtsuka was introduced by Professor A. J. Lohwater.

ALGEBRA AND THEORY OF NUMBERS

526. T. M. Apostol: *Some series involving the Riemann zeta function.*

V. Ramaswami [J. London Math. Soc. vol. 9 (1934) pp. 165–169] has obtained three formulas which can be employed to obtain the analytic continuation of Riemann's zeta function $\zeta(s)$ over the whole s-plane. If $P_n(s)$ denotes the polynomial defined by $n!P_n(s) = s(s+1)(s+2) \cdots (s+n-1)$, then one of these formulas can be expressed as follows: $\zeta(s)(1-2^{-s}-3^{-s}-6^{-s}) = 1+2\sum_{n=1}^{\infty} P_{2n}(s)\zeta(2n+s)6^{-2n-s}$. In this paper simple properties of the Hurwitz zeta function are used to obtain the general identity: $\zeta(s)\sum_{d|k}\mu(d)d^{-s}=\phi_{-s}(k/2;\ k)+2\sum_{n=0}^{\infty}P_{2n}(s)\zeta(2n+s)k^{-2n-s}\phi_{2n}(k/2;\ k)$, where $k\neq 1$ is an integer, $\mu(n)$ is the Möbius function, and $\phi_\alpha(x;k)$ is the sum of the αth powers of those integers not exceeding x which are relatively prime to k. When $k\leqq 6$, $k\neq 5$, the function $\phi_\alpha(k/2;k)$ becomes identically 1 and Ramaswami's formulas occur as the special cases $k=2, 3, 6$. The general formula can also be used to extend some of the other results in Ramaswami's paper. (Received May 7, 1953.)

527*t*. A. L. Foster: *Generalized "Boolean" theory of universal algebras; subdirect sums and normal representation theorem.*

In furtherance of earlier work [*On n-ality theories in rings*, etc., Amer. J. Math. (1950); *p-rings and their Boolean-vector representation*, Acta Math. (1950); etc.], dealing with extensions of the Boolean realm, the author here deals with a broader extension, to a comprehensive variety of universal algebras—of *class f*. The present paper defines and establishes the existence of a *normal expansion* for the functions of such universal algebras $\mathfrak{A}$, and only for such, as are *normal* subdirect sums of an algebra A of class f (=kernel). For appropriate kernels this incorporates the known normal expansions of Boolean-rings, p- and p^k-rings, and Post algebras. Various consequences of the existence of the normal expansion are studied, with emphasis on functionally complete kernels. A universal algebra A is of *class f* if for some e_0, $e_1\in A$ functions x' of the algebra exist such that $'$ is a permutation with $e_0'=e_1$, $e_1'=e_0$, and x is binary with e_0, e_1 as (both sided) null and identity. Rings, complemented lattices, Post algebras, etc., are of class f. A subdirect sum $\mathfrak{A}$ of A is *normal* if $\mathfrak{A}$ contains all $(\alpha, \alpha, \alpha, \cdots)$ $(\alpha\in A)$, and if with $\bar{\alpha}\in\mathfrak{A}$ and for each $\mu\in A$, all projections $P_\mu\bar{\alpha}$ are $\in\mathfrak{A}$. Here with $\bar{\alpha}=(\alpha_1, \alpha_2, \cdots)$, $P_\mu\bar{\alpha}$ replaces α_i by e_1 if $\alpha_i=\mu$ and by e_0 if $\alpha_i\neq\mu$. (Received April 27, 1953.)

528*t*. A. L. Foster: *Identities and subdirect sums of functionally complete universal algebras.* Preliminary report.

Among others the following results are established. *Theorem* 1. *If A and $\mathfrak{A}$ are universal algebras of the same species in which* (1°) *A is finite and functionally complete,* (2°) *every identity of A is an identity of $\mathfrak{A}$,* (3°) *neither A nor $\mathfrak{A}$ is the one-element algebra, then $\mathfrak{A}$ is a normal subdirect sum of A.* This result incorporates the familiar subdirect structure of such algebras as Boolean-rings, p- and p^k-rings, Post algebras, etc. A set I of identities is equationally saturated if I^* has only the trivial one-element algebra as a model; here $I^*=I$ augmented by any equation logically independent of I. *Theorem* 2. *If A is as in Theorem* 1, *the class I of all identities of A is equationally saturated. Moreover I has a finite basis.* A more general result, similar to Theorem 1, is established for the case where A is merely weakly functionally complete, i.e., where constants are permitted in both A-functions and equations. The methods employed borrow heavily from those used in the previous abstract. (Received April 27, 1953.)

529*t*. Mariano García: *On numbers with integral harmonic mean.*

The following results are obtained: (1) If n is an odd integer having an integral harmonic mean for its divisors, the prime factor decomposition of n cannot contain a prime of the form $(4k-1)$ raised to an odd power. (2) No odd number having a prime factor decomposition of the form $p_1^{2^{s_1}-1}\cdot p_2^{2^{s_2}-1}\cdots p_r^{2^{s_r}-1}$ can have an integral harmonic mean for its divisors. (3) Except for perfect numbers, no integer with prime factor decomposition $p^\alpha q$ can have an integral harmonic mean for its divisors.

These results are used to extend the table given by Ore [Amer. Math. Monthly vol. 55 (1948) pp. 615–619] of the positive integers less than 10,000 that have an integral harmonic mean for their divisors. The integers between 1 and 10,000,000 which have an integral harmonic mean are tabulated and these turn out to be all even. (Received May 7, 1953.)

530. J. E. Maxfield and Margaret W. Maxfield: *Sums of powers of numbers having a given period modulo m.*

In this paper the following theorem is proved: The sum of the positive integers less than p^s having period $e=p^re'$ (mod p^s), where $0\leqq r<s$, $p\nmid e'$, $s\geqq 1$, and p is odd, is congruent to $\phi(p^r)\mu(e')$(mod p^s), where $\phi(m)$ is the Euler ϕ-function and $\mu(e')$ is the Moebius function. An analogous theorem is proved for the prime 2 and a solution of the problem modulo any composite is indicated. (Received May 11, 1953.)

531. Ivan Niven: *On the irrationality of trigonometric and hyperbolic functions for rational arguments.*

The method given by the writer to prove the irrationality of π [Bull. Amer. Math. Soc. vol. 53 (1947) p. 509] was extended by others to related problems of irrationality. For example, J. F. Koksma [Nieuw Archief voor Wiskunde (2) vol. 23 (1949) p. 39] treated e^r for nonzero rational r, and R. Rado [J. London Math. Soc. vol. 23 (1948) pp. 267–271] developed a generalization to solutions of certain systems of differential equations, with e^r and tan r as special cases. We here extend the method to cos r and cosh r, from which irrationality of all the trigonometric and hyperbolic functions for nonzero rational arguments follows readily. (Received April 22, 1953.)

532*t*. Alex Rosenberg: *Finite-dimensional and nil subalgebras of certain primitive algebras.*

Let M, N be a pair of dual vector spaces over a division ring D with center Z, and let M and N both be of countable dimensions. Let $A=L(M, N)$ be the ring of all linear transformations on M with adjoints on N and let $S=F(M, N)$ be its socle. The following theorems are proved: Let B be a simple subalgebra of A containing the unit of A and finite-dimensional over Z, then if B' is the commutator of B in A, $B''=B$. If B and C are two isomorphic subalgebras satisfying the same hypotheses as B in the previous theorem, the isomorphism can be extended to an inner isomorphism of A. Let R be a maximal nil subring of S, then if R annihilates a vector in M (R^* annihilates a vector in N) there exists a pair of biorthogonal bases of M, N such that R can be written as the ring of matrices zero on and below (zero on and above) the diagonal. This generalizes a theorem of Shoda-Köthe-Levitzki. (Received May 7, 1953.)

533*t*. Alex Rosenberg and Daniel Zelinsky: *On Nakayama's extension of the $x^{n(x)}$ theorems.*

Nakayama showed (Canadian Journal of Mathematics vol. 5 (1953) pp. 242–244) that if A is a division ring with center Z such that (1): $\alpha_1a^{n_1(a)}+\cdots+\alpha_ra^{n_r(a)}$ is in Z for fixed central elements $\alpha_1, \alpha_2, \cdots, \alpha_r$ and $n_1(a)<n_i(a)$, $i=2, \cdots, r$, then $A=Z$. We show that a primitive ring satisfies (1) if and only if it is commutative or an algebraic algebra over a finite field. In the latter case the center is an absolutely

algebraic field of characteristic $p \neq 0$, and (1) implies the condition (2): $a^{n(a)} - a^{m(a)} = 0$. If A is a semi-simple algebra satisfying (1) with bounded $n_i(a)$ it is a subdirect sum of a commutative algebra and an algebra R. Here R is a subdirect sum of finite simple rings and satisfies (2) with fixed $n(a)$, $m(a)$. If the $n_i(a)$ in (1) are no longer assumed bounded the algebra R need no longer satisfy (2). (Received May 11, 1953.)

ANALYSIS

534. Frederick Bagemihl, Paul Erdös, and Wladimer Seidel: *Some boundary properties of analytic functions defined by certain infinite products in the unit circle.*

It is shown by direct and completely elementary methods that, if the sequence of positive integers $n_j \to \infty$ rapidly enough, $f(z) = \prod_{j=1}^{\infty} \{1 - z^{n_j}(1 - n_j^{-1})^{-n_j}\}$ is regular in $|z| < 1$ and possesses numerous boundary properties, including some established for other functions by Fatou (Bull. Soc. Math. France vol. 48 (1920) pp. 208–314), Lusin and Priwaloff (Ann. École Norm. (3) vol. 42 (1925) pp. 143–191), and others; e.g.: there exists a circle C_n in $|z| < 1$ about each zero of $f(z)$ such that the C_n are mutually exterior to one another and $|f(z)| \to \infty$ as $|z| \to 1$ outside the C_n; $|f(z)| \to \infty$ uniformly on a sequence of circles with the origin as center; $|f(z)| \to \infty$ along almost all radii of $|z| < 1$, and along spirals in $|z| < 1$ which approach $|z| = 1$ asymptotically; $|f(z)| + |f'(z)| \to \infty$ as $|z| \to 1$ in $|z| < 1$; $|f(z)|$ grows arbitrarily slowly as $|z| \to 1$. With suitable modifications of the above product, further examples are obtained concerning the radial limits of functions regular in $|z| < 1$; e.g.: a $g(z)$ for which $|g(z)|$ has no radial limit but $\to \infty$ uniformly on a sequence of circles with the origin as center; an $h(z)$ with $|h(z)| \to 0$ along almost all radii. (Received May 4, 1953.)

535*t*. A. P. Calderón and Antoni Zygmund: *On singular integrals.*

Let $P = (x_1, x_2, \cdots, x_n)$, $K(P)$ a homogeneous function of degree $-n$, $K_\lambda(P) = K(P)$ for $|P| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} > \lambda$ and $K_\lambda(P) = 0$ otherwise, and $f(P) \in L^p$. The authors showed in a previous paper that if $K(P)$ satisfies certain continuity conditions, then the integral $\int_{E_n} K_\lambda(P - Q) f(Q) dQ$ converges in the mean of order p and pointwise almost everywhere as $\lambda \to 0$. The continuity conditions on $K(P)$ are removed and these results are shown to hold if: (i) $K(P)$ is odd, that is, $K(P) = -K(-P)$, and the restriction of $|K(P)|$ to the spherical surface $|P| = 1$ is integrable; (ii) $K(P)$ is even, that is, $K(P) = K(-P)$, and the restriction of $|K(P)| \log^+ |K(P)|$ to $|P| = 1$ is integrable. These results are the best possible of their type and are obtained by a method entirely different from the previous one. (Received April 29, 1953.)

536. C. A. Hayes and C. Y. Pauc: *The problem of the differentiation of set functions.* Preliminary report.

This paper complements the papers *Vitalische Systeme in booleschen σ-Verbanden* by O. Haupt and C. Y. Pauc, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Klasse der Bayerischen Akademie der Wissenschaften zu München (1950) pp. 187–207, *Differentiation with respect to φ-pseudo-strong blankets* by C. Hayes, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 283–296, and *Differentiation of some classes of set functions* by C. Hayes, Proc. Cambridge Philos. Soc. vol. 48 (1952) pp. 374–382. The writers treat several types of differentiation bases (of which *blankets* are a special case) endowed with various properties which are sufficient to permit the differentiation of suitable corresponding classes of set functions. The present paper deals with the converse problem; it is shown for each of the different types of bases that the possession of differentiation power implies a corresponding covering property. The relationship between the present paper and some earlier efforts of R. de Possel, H. Busemann, W. Feller, and B. Younovitch are also explored. (Received April 6, 1953.)

537*t*. V. E. Hoggatt, Jr.: *The inverse Weierstrass p-function: numerical solution and related properties.* Preliminary report.

The inversion integral for the Weierstrass elliptic p-function is numerically evaluated by expanding the integrand in a power series and applying the contour integral term by term. The only assumption is a knowledge of the periods ω and ω' of $p(z)$. If $g_2\neq 0$, the transformation $g_2=12\lambda^4$; $g_3=4(2+h)\lambda^6$; $t=\lambda^2x$ reduces the expression under the radical in the integrand from $(4t^3-g_2t-g_3)$ to $[4\lambda^6\{x^3-3x-(2+h)\}]$ which is expanded in a Maclaurin's expansion in h. For $p(z)=0$ the series converges absolutely and uniformly for $|h|<2$. The resulting contour integrals, which are the coefficients of powers of h, are reducible to a finite number of elementary functions. The series is valid for finding $z=p^{-1}(\alpha)$ for all complex α except in three regions around the singularities of the coefficient integrals and applies to the entire rectangular case and a large part of the skew case. (Received May 6, 1953.)

538*t*. R. E. Lane: *Note concerning the Stieltjes integral.*

A modification (based on the trapezoidal rule) of the definition of the Riemann-Stieltjes integral is used to define a more general integral which has most of the desirable properties of the Stieltjes integral, including integration by parts. Under this definition the integral $\int_a^b u(x)dv(x)$ exists if, for example, u is *any* function which has only discontinuities of the first kind in $[a, b]$ and v is *any* function which is of bounded variation in $[a, b]$. The definition can be extended to functions of more than one variable. (Received April 8, 1953.)

539. R. B. Leipnik and A. P. Morse: *Heaviside calculus of difference operators.*

The Heaviside calculus of power fields of operators developed by the authors is applied to the simple special case of difference operators. Strengthened forms of classical results of difference calculus are obtained with ease. (Received June 20, 1953.)

540*t*. Makoto Ohtsuka: *Capacity of product sets.*

Let Ω be a metric space and $U_\alpha^\mu(P)$, $\alpha\geqq 0$, the potential $\int_\Omega\phi(r_{PQ})\,d\mu\,(Q)$, where $\phi(r_{PQ})=r_{PQ}^\alpha$ if $\alpha>0$ and $\phi(r_{PQ})=-\log r_{PQ}$ if $\alpha=0$, and where μ is a non-negative distribution of unit mass on a bounded closed set F in Ω. Let $V=\inf_\mu(\sup_{P\in\Omega}U_\alpha^\mu(P))$, and define $C^{(\alpha)}(F)=\phi^{-1}(V)$. For a set X in Ω, define $C_i^{(\alpha)}(X)=\sup C^{(\alpha)}(F)$ over all closed bounded sets $F\subset X$. A bounded set X is called an α-polar set if there exists μ such that $U_\alpha^\mu(P)=\infty$ for every $P\in X$. Let $X\times Y$ be a product set in the metric space $\Omega_1\times\Omega_2$, where $X\subset\Omega_1$, $Y\subset\Omega_2$. Then a lower bound for $C_i^{(\alpha+\beta)}(X\times Y)$ is given in terms of $C_i^{(\alpha)}(X)$ and $C_i^{(\beta)}(Y)$. If Ω_1 is Euclidean n-space, then the lower bound for $C_i^{(n+\beta)}(X\times Y)$ can be expressed in terms of $C_i^{(\beta)}(Y)$ and $m_n(X)$, the n-dimensional inner Lebesgue measure of X. It is shown that $C_i^{(n+\beta)}(X\times Y)>0$ if $m_n(X)>0$ and $C_i^{(\beta)}(Y)>0$, and that $X\times Y$ is an $(n+\beta)$-polar set if X is bounded and if Y is a β-polar set. Applications are made to the regularity criteria of boundary points for the Dirichlet problem. (Received May 6, 1953.)

541. R. G. Selfridge: *Generalized Walsh transforms.*

Complex-valued Walsh functions have been introduced by Chrestenson (see Bull. Amer. Math. Soc. Abstract 59-4-503) and shown to form an orthonormal, complete system over [0, 1]. For the complex-valued Walsh function one can define a Walsh function with arbitrary subscript by $\psi_y(x) = \psi_{[y]}(x)\psi_{[x]}(y)$. The resulting class includes the ordinary Walsh functions (see Fine, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 66–77). With this Walsh function one can develop a transform theory for functions in $L_p(0, \infty)$. For functions in L_2 a Plancherel theorem is developed. For functions in L_p, $1<p<2$, a transform exists that maps L_p into $L_{p/(p-1)}$. If $f(x)$ is in L_1 and $F(y) = \int_0^\infty f(x)\psi_y(x)dx$, then many of the theorems of ordinary Fourier transforms carry over. In particular one has $f(x) = \lim_{R\to\infty}\int_0^R F(y)\overline{\psi_x(y)}dy$ if $f(y)$ is of bounded variation in a neighborhood of x and x is either a point of continuity or a point of right-hand continuity and has a finite expansion in powers of α, where $\psi_n(x)$ has been developed in terms of α. Further one has that $f(x) = \lim_{R\to\infty}\int_0^R(1-[y]/R)F(y)\overline{\psi_x(y)}dy$ for almost every x at which $f(y)$ is locally essentially bounded. (Received April 20, 1953.)

542t. R. N. Tompson: *Areas of k-dimensional non-parametric surfaces in k+1 space.*

Suppose k is a positive integer greater than one, X is a k cell in Euclidean k space E_k, g is a continuous real-valued function on X, and the function f is the k-dimensional nonparametric surface in E_{k+1} defined by $f(x) = (x_1, \cdots, x_k, g(x))$ for $x \in X$. It is shown that the k-dimensional Lebesgue area of f = the k-dimensional integralgeometric stable area of $f = F^k_{k+1}$ (range f), where F^k_{k+1} is the k-dimensional integralgeometric (Favard) measure over E_{k+1} (see H. Federer, *Measure and area*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 306–378). It is also proved that F^k_{k+1} (range f) is finite if and only if g is of bounded variation in the sense of Tonelli. If g is of bounded variation in the sense of Tonelli, then $\int_X Jf(x)dL_k x \leqq F^k_{k+1}$ (range f) where Jf is the Jacobian associated with f by means of its approximate differential and L_k is the k-dimensional Lebesgue measure. Equality holds if and only if g is absolutely continuous in the sense of Tonelli. (Received March 17, 1953.)

543t. Alexander Weinstein: *On the Cauchy problem for the Euler-Poisson-Darboux equation.*

Let k be a real parameter, $k \neq -1, -3, \cdots$, and let x denote a system of m variables $x_1, \cdots, x_m$. Let $u^k(x, t)$ satisfy the equation (1) $\Delta u^k = u^k_{tt} + kt^{-1}u^k_t$. A solution of the Cauchy problem with the initial data $u^k(x, 0) = f(x)$, $u^k_t(x, 0) = 0$, given in a previous paper (C. R. Acad. Sci. Paris vol. 234 (1952) pp. 2584–2585) is replaced by a new and simpler formula which is valid for $k < m-1$. Let the integer n satisfy the inequality $k+2n \geqq m-1$. By the results obtained previously, u^{k+2n} can be determined by the initial conditions $(k+1) \cdots (k+2n-1)u^{k+2n}(x, 0) = f(x)$, $u^{k+2n}_t(x, 0) = 0$. A repeated application of the formulas $u^k = t^{1-k}u^{2-k}$, $u^k_t = tu^{k+2}$ yields the new formula $u = t^{1-k}(\partial/t\partial t)^n(t^{k+2n-1}u^{k+2n})$ which solves Cauchy's problem for $k < m-1$. Another solution of the same problem which includes the exceptional values $k = -1, -3, \cdots$ will be published by Diaz and Weinberger in the Proceedings of the American Mathematical Society. This work has been sponsored by the Office of Naval Research. (Received May 6, 1953.)

Applied Mathematics

544t. Abraham Charnes: *Constrained games and linear programming.*

Constrained games are viewed as "semi-normalization" of unconstrained games. Equivalence of optimal play for player 1 vs. player 2 with (resp.) dual linear programming problems is established in a fixed-point-free proof of the saddle-point property of a pair of dual optimal solutions. Fixed-point-free proof of the fundamental theorem for convex games is a corollary. (Received April 30, 1953.)

545. F. E. Ehlers: *The induced velocity field of an axially symmetric supersonic jet at an angle of attack in a supersonic stream.* Preliminary report.

The linearized equations for the perturbation velocity potentials inside an axially symmetric jet and outside in the supersonic stream are solved by the method of the Laplace transform. The angle of incidence and the pressure difference are assumed small. As a consequence the conditions of continuous pressure and velocity inclination which must hold on the free boundary of the jet are satisfied instead on a circular cylinder which nearly represents the jet. The results are obtained in the form of infinite integrals which can be evaluated numerically. (Received May 4, 1953.)

546t. P. G. Hodge, Jr.: *The effect of strain-hardening on an annular slab.*

A procedure is outlined for obtaining the stresses and strains in a circular slab with a cutout, subject to uniform biaxial tension. An arbitrary stress-strain curve in tension is approximated by any number of straight-line segments. For biaxial states of stress the material is assumed to satisfy a flow law based on the maximum shear stress, and to be incompressible throughout. The general equations are given and then simplified by assuming that boundary motions can be neglected if the strains are small, and that elastic strain components may be neglected if the strains are large. For the case of linear strain-hardening a complete solution is given in closed form. If the rate of strain-hardening is small, these results may be further simplified. (The results presented in this paper were obtained in the course of research conducted as a Consultant under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received May 6, 1953.)

547t. Eugene Levin: *Symmetric reinforcements of circular cutouts.*

A slab with a circular cutout and a reinforcement of arbitrary symmetric cross section is subjected to stresses in the plane of the slab. The slab and the reinforcement are assumed to be of a perfectly plastic material which satisfies Tresca's yield criterion. Under the assumption of generalized plane stress and using a limit design theorem of Drucker, Greenberg, and Prager (Quarterly of Applied Mathematics vol. 9 (1952) pp. 381–389) a method is described for the determination of a lower bound on the collapse load. The results agree with those previously obtained by Weiss, Prager, and Hodge (Journal of Applied Mechanics vol. 19 (1952) pp. 397–402) for the special case of a cylindrical reinforcement. (The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received May 6, 1953.)

548. W. M. Stone: *On the effect of a randomly modulated carrier in pulsed radar theory.*

J. I. Marcum (Report 314, Rand Corporation) has exhaustively discussed the probability of detection for a pulsed radar, assuming a constant amplitude of signal return from target. The theory has been extended to include a randomly modulated signal by means of an averaging process. A considerable number of curves are constructed by using either an Edgeworth asymptotic expansion or by the numerical evaluation of a certain definite integral. (Received May 4, 1953.)

GEOMETRY

549t. H. W. E. Schwerdtfeger: *On Jacobsthal's circle matrices.*

Let K be a matrix with complex elements a and b in the first row and c and d in the second row. It is said to be a circle matrix (E. Jacobsthal, *Zur Theorie der linearen Abbildungen*, Berliner Math. Ges. Sitzungsber. vol. 33 (1934) pp. 15–34) if it satisfies the condition $K\overline{K}=\mu E$ (E=a matrix with 1 and 0 in the first row and 0 and 1 in the second row). This is necessary and sufficient for the transformation $Z=(a\bar{z}+b)/(c\bar{z}+d)=K(\bar{z})$ to be an inversion, with respect to a real circle (denoted by $\mathfrak{K}$) if the multiplier $\mu>0$, with respect to an imaginary circle if $\mu<0$. If $c=0$ the circle $\mathfrak{K}$ becomes a straight line; if $\mu=0$, a point circle. The equation of the circle $\mathfrak{K}$ is given by $z=K(\bar{z})$. Jacobsthal suggests (loc. cit. p. 28) the study of the circles $\mathfrak{K}^m$ for all integral m which corresponds to the circle matrices K^m. Instead we consider the continuous iteration $\mathfrak{K}^s$ for all real s, corresponding to the iteration K^s of the Moebius transformation K, i.e. $Z=K(z)$. It is found that for $\mu>0$ the transformation K can only be either hyperbolic (then the circle $\mathfrak{K}$ has no point in common with the real axis) or parabolic (then the circle $\mathfrak{K}$ touches the real axis at the fixed point of the Moebius transformation K) or elliptic (when the circle $\mathfrak{K}$ cuts the real axis at the fixed points of the transformation K); further that the circle $\mathfrak{K}$ is an element of the pencil P of all circles which are interchanged by the transformation K, thus orthogonal to the invariant circles. Then it is shown that the continuous iteration K^s $(-\infty<s<+\infty)$ corresponds to the continuous development of the pencil P out of the circle $\mathfrak{K}$. (Received May 21, 1953.)

STATISTICS AND PROBABILITY

550t. Ernest Ikenberry: *The probability of convergence of Gram-Charlier expansions of distribution functions.*

This paper gives the probabilities of convergence of Gram-Charlier expansions exp. $\{-h^2(x-X)^2\}\cdot\sum_{r=0}^{\infty}C_rH_r\{h(x-X)\}$ of the Gaussian distribution function $f(x)=((1/(2\pi)^{1/2})\sigma)\cdot\text{exp.}\ \{-(x-\mu)^2/2\sigma^2\}$, where X (the mean) and $2/h^2=s^2$ (the variance) are evaluated for a random sample $\{X_i\}$ of n from a normal parent population $N(\mu,\sigma^2)$ with mean μ and variance σ^2. It has recently been shown that the series expansion exp. $(-h^2\omega^2)\cdot\sum_{r=0}^{\infty}a_rH_r(hw)$ of exp. $(-\alpha w^2)\cdot P(w)$, where the $H_r(hw)$ are Hermite polynomials of degree r and $P(w)$ is any entire function of order less than 2, or of minimal type if of order 2, converges provided that h^2 lies inside of the circle $|h^2-\alpha|=|\alpha|$ in the h^2-plane [E. Ikenberry and W. A. Rutledge, Journal of Mathematics and Physics vol. 31 (1952) pp. 180–183]. Since the computed h^2 is always real and positive, the probability of the convergence of the expansion of $f(x)$ is given by $P=P(h^2<1/\sigma^2)=P(\nu s^2/\sigma^2<\nu/2)$, where $\nu=n-1$ is the number of degrees of freedom. It is known that $(\nu s^2/\sigma^2)$ is distributed as χ^2 with $\nu=n-1$ degrees of freedom. Hence P, for a given ν, may be obtained by locating in a table of percentage points of the chi-square distribution the tabular value of χ^2 which is equal to $\nu/2$, interpolating where necessary. Values of P range from .342 for $\nu=1$ to .988 for $\nu=29$. (Received March 20, 1953.)

551*t*. Jacob Wolfowitz: *Estimation of a linear relationship when both variables are subject to error.*

The estimators obtained by the author (Skandinavisk Aktuarietidskrift, 1952, pp. 132–151) converge with probability one no matter what the covariance matrix of u and v is. Under weak regularity conditions the results are valid when the incidental parameters are constants (and not chance variables). When the incidental parameters are independent, identically distributed chance variables, these regularity conditions are fulfilled with probability one. In the notation of the paper cited let $D_n(x)$ be a normal distribution function with mean zero such that $\delta(B_n(x), D_n(x))<1/n+\delta(B_n(x), N^*)$. The variance of $D_n(x)$ converges to the variance of $(v-\beta u)$ with probability one. (Received April 30, 1953.)

552*t*. Jacob Wolfowitz: *The estimation of structural parameters.*

Let $\{X_{ij}\}$, $i=1, \cdots, n$; $j=1, \cdots, m_i$, be independently distributed chance variables. Let $F_i(x|\theta, \alpha_i)$ be the distribution function of $X_{i1}, \cdots, X_{im_i}$, and $B^n(x)$ be the empiric distribution function of $\{X_{ij}\}$, for all i and j. Let $C^n(x|\theta', \alpha')$ be $(\sum_{i=1}^n m_i)^{-1} \sum_{i=1}^n m_i F_i(x|\theta', \alpha_i')$. Let $\theta_n^*, \alpha_{1n}^*, \cdots, \alpha_{nn}^*$ be Borel measurable functions of $\{X_{ij}\}$ such that, writing $\alpha_n^*=(\alpha_{1n}^*, \cdots, \alpha_{nn}^*)$, $\sup_x |C^n(x|\theta_n^*, \alpha_n^*)-B^n(x)|<1/n+\inf_{\theta',\alpha'} \sup_x |C^n(x|\theta', \alpha')-B^n(x)|$. Under mild regularity conditions θ_n^* converges to θ with probability one. (Received April 30, 1953.)

Topology

553*t*. B. J. Ball: *A note on paracompactness.*

It is shown that every linearly ordered space is countably paracompact and that a necessary and sufficient condition that a locally compact linear space be paracompact is that it be covered by a collection of mutually exclusive, closed, open sets each of which is the sum of a countable number of intervals. A sufficient condition that a normal Hausdorff space H be paracompact is that it have the Lindelöf property and in case every uncountable subset of H has a limit point, this condition is also necessary. (Received May 7, 1953.)

554*t*. Armand Borel: *The ends of homogeneous spaces of Lie groups.*

G denotes a connected Lie group, H a closed subgroup of G. It is first proved that if H is connected, the homology groups of the coset space G/H are isomorphic to those of K/L, where $K\supset L$ are maximal compact subgroups of G and H respectively; from that and from a theorem of E. Specker one deduces that if H is connected, G/H has at most two ends (in the sense of Freudenthal). It is moreover established that if G/H has two ends, it is homeomorphic to the topological product of K/L by a line. Simple 3-dimensional examples (manifolds of unit tangent vectors to certain Riemann surfaces) show the existence of homogeneous spaces G/H (H non connected) having an infinite number or any finite number of ends. The second part of the paper deals with the determination of spaces which are called m-homogeneous, i.e. which admit a Lie group of homeomorphisms transitive on m-tuples of points. The results are as follows: There is no simply connected space which is either non compact and m-homogeneous for $m \geqq 3$, or compact and m-homogeneous for $m \geqq 4$. The simply connected 2-homogeneous spaces are the euclidean spaces, the spheres, the complex and quaternionian projective spaces, the projective plane over the Cayley numbers, and, among them, only the spheres are 3-homogeneous. (Received May 4, 1953.)

555*t*. S. D. Liao: *On topology of cyclic products of spheres.*

The p-fold cyclic product Σ_{np} of an n-sphere S_n, $n \geqq 2$, has vanishing integral cohomology groups $H^s(\Sigma_{np}, Z)$ for $1 \leqq s < n$, $s = n+1$, and has $H^n(\Sigma_{np}, Z) \approx Z$. We calculate for a prime p the iterated cyclic reduced powers (mod p) of the generator of $H^n(\Sigma_{np}, Z)$. We study also some homotopy properties of Σ_{np} for $p=2$, $p=3$. We have that the 2-primary subgroup of $\pi_s(\Sigma_{n2})$ vanishes for $5 \leqq n < s \leqq n+3$. Let S_n be oriented, and let it be imbedded in Σ_{np} for an arbitrary p such that S_n represents the generator of $H_n(\Sigma_{np}, Z)$. Let E_{n+4} be an oriented $(n+4)$-cell. Then, a representative map $f: \partial E_{n+4} \rightarrow S_n$ of an element e of $\pi_{n+3}(S_n)$ extends over E_{n+4} in Σ_{n2} for $n \geqq 5$ if e is a 2-primary element and in Σ_{n3} for $n \geqq 3$ if e is a 3-primary element. In the former case, $\pi_{n+3}(S_n)/2\pi_{n+3}(S_n)$ is characterized by the square operation Sq^4. In the latter case the 3-primary subgroup of $\pi_{n+3}(S_n)$ is characterized by the reduced power P^1(mod 3). These results are useful for the study of fourth obstructions of sphere bundles. (Received May 18, 1953.)

556*t*. R. F. Williams: *Local contractions of compact metric sets which are not local isometries.*

The purpose of this paper is to answer a question raised by A. Edrei in a paper entitled *On mappings which do not increase small distances*, which appeared in the September 1952 issue of the Proceedings of the London Mathematical Society. The author demonstrates the existence of a compact subset of the plane M and a non-local-isometric mapping f of M onto M such that each point of M is a point of contraction under f (thus f might be called a local contraction). It is further pointed out that minor alterations of M and f give such examples where f is a homeomorphism and (1) M is in the plane, (2) M is a connected subset of Euclidean three-space, (3) M is totally disconnected. (Received May 4, 1953.)

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