to satisfy myself that either (a) the statement is false, or (b) I do not understand what it means.

I believe that there is a rather bad error here, which the author would do well to correct if he gets the chance in a later edition—maybe he will have to do quite a bit of rewriting. But even if the error is corrected, that does not answer the question: Shall we impose mathematics by authority? My own view is that we should not. Authoritative statements cannot be completely avoided, but they should be supported by plausibility-arguments and by the working out of special cases within the scope of the reader. This takes space, but it is space well used if one thereby establishes confidence and a sense of reality. In the last analysis, it is the special case that establishes confidence, not the general theorem, and this holds for everyone, high and low.

J. L. SYNGE

Les nombres inaccessibles. By É. Borel. Paris, Gauthier-Villars, 1952. 10+141 pp. $3.72.

The author prefaces the work under review as follows:

"This little book is the result of half a century of reflections on the principles of mathematical analysis and, in particular, on the definition of numbers. Some of these reflections have already been sketched here and there in the works of this Collection, but it seemed to me that it would be useful to coordinate them in a connected account.

"The profound transformations of physics in the twentieth century, and especially the theories of relativity, quanta, and wave mechanics, have been inspired by the fundamental idea that phenomena must be observed *en eux-mêmes*, without taking account of *a priori* conceptions such as time, space, matter, or energy—conceptions with which one has associated absolute and immutable entities.

"It seems to me that mathematicians as well, while maintaining the full right to work out abstract theories deduced from arbitrary noncontradictory axioms, have an interest in distinguishing, among the objects of thought which are the substance of their science, those which are truly accessible, that is to say, have an individuality, a personality, which characterizes them without ambiguity. One is thus led to define in a precise manner a science of the accessible and of the real, beyond which it remains possible to develop a science of the imaginary and of the imagined, these two sciences being able, in certain cases, to lend each other mutual support.

"Such is the spirit in which I have written this book, which I submit to the reflections of the young mathematicians whose efforts will contribute to the orientation towards perpetually new developments, of the science which is the starting point of all the others, and which will undoubtedly continue for a long time to be the vital source whence the progress of our knowledge and the perfection of our techniques will spring."

Borel's thesis is that the overwhelming majority of numbers will always remain inaccessible to the human race as we know it, in the sense that it will never be possible to define these numbers effectively in such a manner that any two mathematicians will be certain that they are speaking about one and the same entity. Indeed, some integers of one hundred digits, even if written down, will never be "known," because of the human impossibility of discovering any but trivial consequences of their definition. The boundary between the accessible and the inaccessible, however, is itself inaccessible. The properties of a set whose elements are inaccessible can be studied—even though it is impossible to study the elements individually—but only by means of the calculus of probabilities.

The author is thus naturally led to discuss systems for representing numbers, methods of defining numbers, enumerable sets and the continuum in relation to measure and probability, and the axiom of choice.

Many parts of the book, e.g. the chapter on systems of numeration, are delightful because of interesting remarks which lend a surprisingly new aspect to simple, familiar mathematical objects. The somewhat novelistic style of writing makes superficial reading easy, penetrating reading more difficult. A definition, for instance, given merely by analogy with some previously described examples is not an aid to understanding an inherently difficult subject, and may actually serve to cast suspicion on the clarity of the motivating idea.

Readers will find some of Borel's assertions provocative, and this is probably one of the chief merits of the book: it contains material for thought and discussion. Several statements in the above-quoted preface alone, e.g., the comment about physics, would serve (and, in fact, have served the author himself) more appropriately as subjects for another book rather than for brief evaluation in a book review. We shall therefore leave aside for the most part the broader philosophical aspects of the volume. The ensuing remarks are devoted mainly to pointing out what appear to be some specific inconsistencies.

Borel infers from the homogeneity of the continuum that two equal subintervals of the unit interval are indiscernible, and that therefore the probability that a point chosen at random in the unit interval fall in one subinterval is equal to the probability that it fall in the other, and, further, is equal to the length of the subinterval. It is difficult to see how he can consider two equal intervals indiscernible if, say, the left end point of one is accessible and that of the other is inaccessible. Nor does he give adequate reason for assigning the same probability to two equal subintervals if, e.g., one is near the origin and the other is near the center of the unit interval, especially in view of the fact that he interprets choice and probability in the naïve sense.

If an enumerable set, such as that of the natural numbers, is considered instead of the unit interval, then the assignment of equal probabilities to the elements of this set reduces to zero the global probability of any number of accessible integers, which, according to Borel, is absurd because it precludes the possibility of ever getting one of these numbers, so that every choice leads to an inaccessible number. This interpretation of probability zero as an expression of impossibility is, however, not justified, and the author conveniently avoids it when admitting that the accessible points of the unit interval have probability zero.

Borel proposes to assign a *positive* probability to *each* element of an enumerable set in such a manner that these probabilities form a convergent series whose sum is unity. He admits that no simple and natural method for doing this presents itself, but argues that there are many arbitrary conventions for making such an assignment (e.g., one under which probability 1/2 is assigned to the set of accessible integers as well as to the set of inaccessible integers) and that, in many problems, which particular assignment is selected does not influence the conclusion drawn. Now in many applications of Zermelo's axiom of choice it is also true that the conclusion drawn does not depend on the particular choice function, but Borel ignores this fact: He interprets Zermelo's axiom as affirming that it is possible to choose (in the literal sense) a definite number from, e.g., the set of inaccessible integers, so that if this number is denoted by a, then a designates a well-determined number, the same for all mathematicians. He then regards this as a meaningless operation because a is not, and never will be, distinguished from the other inaccessible numbers. It is not clear why he does not regard the assignment of a positive probability to every inaccessible a, or of probability 1/2 to the set of inaccessible integers, as an equally meaningless operation, inasmuch as the set of inaccessible integers is no more well-determined than the a referred to above in connection with Zermelo's axiom.

Borel discusses the familiar decomposition of the circumference of a circle into an enumerable number of mutually exclusive, congruent sets of points. He asserts that it is not possible to attribute equal probabilities to these sets without running into contradictions, and that it is therefore necessary to attribute unequal probabilities to them. "But then we contradict the Euclidean principle of equality, according to which two superposable figures are equal." As the construction of these sets "requires the use of Zermelo's axiom, our conclusion is that it is necessary to choose between Zermelo's axiom and Euclid's axiom according to which two superposable figures are equal, that is to say, identical from all points of view, and that, in particular, equal probabilities correspond to them. The simultaneous application of the two axioms leads, in fact, to a contradiction." (Borel, needless to say, chooses "Euclid's axiom.") This argument is open to objections. First, it is not inconceivable that a nonmeasurable set can be constructed without the intervention of Zermelo's axiom. Secondly, there is another way in which two superposable figures may be "identical from all points of view," without having equal probabilities correspond to them, and that is, by having *no* probabilities correspond to them. Euclid's axiom cannot be interpreted as stating that congruent figures—if the "figures" in question are not the elementary Euclidean ones—*have probabilities* and that these probabilities are equal. The author seems to be taking account here of just such an *a priori* conception—probability—as he implies opposition to in his preface.

It is undoubtedly valuable to have in one volume the ideas and opinions of so famous a mathematician on such an important and controversial subject as the foundations of analysis. The history of mathematics shows, however, that what is considered real or imaginary is not an absolute concept, but is relative to the development of our knowledge, and furthermore, that "abstract theories" often throw sufficient light on objects of thought to transfer them from the realm of the imaginary to that of the real. Complex numbers were once considered meaningless, whereas today some mathematicians consider Zermelo's axiom and its consequences meaningless. Borel's notion of accessibility, although of heuristic significance, seems too subjective, temporal, and, by precluding intrinsically the possibility of delimiting the realm of the accessible, vague, according to his own standards, to "define in a precise manner a science of the accessible and of the real."

F. BAGEMIHL