author cites as a noteworthy phenomenon an example of a continuous game where every pure strategy is employed in the optimum strategy. Such examples are commonplace in the statistical literature; several interesting ones (designed for another purpose) are to be found in Ann. Math. Statist. (1950) p. 190. Other citable results are those on the equivalence of behavior strategies and mixed strategies under general conditions (Ann. of Math. (1951) p. 581), and non-trivial results on the elimination of randomization (Ann. Math. Statist. (1951) p. 1 and p. 112).

The above criticisms should be regarded as directed at minor blemishes of a highly meritorious piece of work. The mathematical public is indebted to the author for an excellent and highly readable book, which this reviewer read with pleasure.

J. WOLFOWITZ

Leçons d'analyse functionelle. By F. Riesz and B. Sz.-Nagy. Budapest, Akadémiai Kiadó. 8+448 pp. About $7.50.

This work is superb.

For the field which it covers, it cannot be approached now nor will be soon by other books. It is not presented as a treatise for specialists, the essential purpose of which is to report advanced and complex results. Nor is it written as a textbook for the young student. Its aims are much higher and much more elegant. And in accomplishing these aims its authors have put together a magnificent advanced treatise and a most excellent though not elementary text. The purpose of the work is to set down, within the spirit and context of the undertaking, a certain coherent and central portion of mathematics in final and definite form. And within the spirit of the undertaking, this version is final and correct. Whether it is the only possible such version is another question, the answer to which is not important at this point. The hallmark of the work is its balance and good taste: in the choice of subjects, in the extent and detail in which they are developed, in the methods used to present them, and in the critical question of style and exposition.

The subjects treated are the modern theory of integration and differentiation, and the theory of linear operators which is based upon these concepts. Thus we find discussion of the space L^2 of square integrable functions, of abstract Hilbert space, of the space C of continuous functions. The latter is connected to integration theory by the fundamental correspondence between linear functionals and measures. This leads to a brief treatment of the spaces L^p, $p \geqq 1$, of reflexive spaces, and finally, of Banach spaces. For these various spaces, the operator theory discussed includes integral operators, completely continuous operators in general, completely continuous symmetric (that is, self-adjoint) operators, bounded symmetric operators, unbounded self-adjoint operators, and spectral theory in general Banach spaces.

The method of presentation is a mixture of the inductive and the axiomatic. Thus integration is first developed for the line, then in n-space, and finally abstract integrals and measures are treated. Similarly, in the realm of normed spaces, there is first a thorough treatment of L^2, then the Hilbert space is axiomatized. After discussion of L^2, L^p, $p \geqq 1$, and C, a general Banach space is introduced. The first operators handled are integral operators. From there one generalizes to the completely continuous ones. For the symmetric case there is the succession of stages of generalization mentioned in the preceding paragraph. This method has a multitude of advantages. The historical development of a subject is always humanistically rich. It so happens that in this field, a historical development written in full knowledge of what we now understand is thoroughly satisfactory. This is not to say that the authors insist uniformly on the time-sequence procedure. For instance, at the very beginning, the order of discussion is: differentiation, integration, and measure,—a precise about-face of the traditional sequence. The inductive development has disadvantages which will sometimes discourage the younger student. Thus after he has labored assiduously studying completely continuous operators in L^2, finally establishing the validity of Fredholm's alternative, he is told in one page how to change all the arguments so far presented to apply to a general Banach space. This amounts to receiving a commission to rewrite ten pages of the book to suit the new exigencies. It is doubtful whether these instructions will convince the reader who meets this situation for the first time.

The book is in two parts. The first one on modern variables covers approximately one third of the work. We are told that this section was written by F. Riesz and that Part II on integral equations and linear operators was essentially set down by B. Sz.-Nagy. The appearance finally of Riesz' treatment of real variables is a long awaited event which we all greet with joy. The reviewer has heard many stories of earlier manuscripts which were all ready twenty and more years ago but which due to an exceptional (and exasperating) desire for perfection never reached the printer. A recent inspection of a note-book that this writer kept of daily conversations with F. R. reveals that the treatment of real variables discussed a generation ago is precisely the one now before us. Here it is in the exceptional style and the warm and particularly personal French of its sponsor. The presentation of the entire theory in 140 pages is a remarkable tour de force both of mathematical profundity and of stylistic power. Part II was written by Nagy although it is very heavily influenced by Riesz. The presentation of the junior member of this partnership is of the finest. A more capable and polished writer could scarcely be found. We find in him the same infinite care for smooth and natural development which characterize his teacher. If a difference is to be pointed out, it would be that Nagy is less likely to "épater" the reader with a mild looking but in truth devastating lemma than his kindly malicious preceptor.

Before discussing the substance of the work, a few more general remarks may be made. The mechanics of subdivision within chapters is reduced to a minimum—theorem 3.1.7.12 is nowhere to be found. Instead we have Lebesgue's theorem or Fubini's or Beppo Levi's, etc. This is very satisfactory to the reader, although as a device, it is frequently employed (by other authors) to torture the facts. There is a most useful bibliography citing the works of about one hundred and fifty authors. The errata are infrequent and will cause no difficulty to the reader. Some of these are listed on the last page. The typography is most agreeable. It is a pleasure to be able to report that it is possible to import the book into this country. Some thirty copies were obtained after a delay of three months for a class in integration now being given by this writer.

Part I treats in that order: differentiation, integration, measure; the space L^2; and the Lebesgue-Stieltjes integral. It begins with a three page proof of Lebesgue's theorem on the differentiability almost everywhere (a.e.) of a monotone function. Then comes Fubini's important theorem on the termwise differentiability of a series whose terms are monotone functions of the same type. This theorem yields numerous applications. Next is a section on functions of intervals, their integrals and derivatives. After two brief fundamental theorems for these, it is easy to establish that: a bounded function $f(x)$ is Riemann integrable if and only if it is continuous a.e.

The integral is next defined. First for the class C_0 of step functions in an obvious way. Then for the limits of increasing sequences of step functions whose integrals are bounded—this gives class C_1. Class C_2 is the set of all functions $f(x)-g(x)$ where $f(x)$, $g(x)\in C_1$. This is a very characteristic device with a large number of possible variations and applications. The point of this development is that it incorporates into a definition one of the most important properties of the integral: The integral of a limit is the limit of the integrals. Next comes the theorem (Beppo Levi) that class C_3 (limits of increasing functions in C_2 whose integrals are bounded) is identical with C_2. Essentially of the same character are the classic theorem of Lebesgue and a lemma of Fatou. This completes the development of the integral (in ten pages). It is now possible to develop the inequalities of Schwarz, Hölder and Minkowski fundamental to the study of L^2 and L^p. Finally measurable functions are defined and then measurable sets. A measurable function is one which is the limit a.e. of a sequence of integrable functions. A measurable set is one such that its characteristic function is measurable. The properties of measurability are dealt with in two pages.

Now comes the study of indefinite integrals and their derivatives. With the help of Fubini's theorem (above) it is trivial to prove that every integrable function is the derivative a.e. of its indefinite integral. Using the concept of absolute continuity, the characterization of all indefinite integrals is then taken up.

There is now a section on the space L^2. The well known method of Riesz is applied to the proof of the Riesz-Fisher theorem. Linear functionals and orthonormal systemsare studied. The notions of strong and weak convergence are introduced. In germ form one finds here a multitude of phenomena and theorems which will subsequently be reproduced in Banach spaces (for example: The sequence $\{A_n\}$ of linear functionals in L^2 cannot converge for every $f(x) \in L^2$ without being uniformly bounded).

Functions of several variables may be treated by an obvious generalization or by the following mapping procedure. Suppose that we consider two variables. In the theory of integration for the line it is sufficient to consider step functions which are constant on intervals of the form $m3^{-n} \leqq x \leqq (m+1)3^{-n}$; similarly for squares. Now one may establish a 1-1 correspondence between the 3^{2n} squares each of area 3^{-2n} of the unit square and the 3^{2n} intervals each of length 3^{-2n} of the unit interval. This correspondence may be established for each n in the customary way so as to "preserve inclusion." This process now yields a 1-1 correspondence between step functions in two dimensions defined over the planar grid and those in one dimension defined over the linear grid. This correspondence preserves integrals. It may be extended to limits of increasing sequences of step functions. With the help of this correspondence many problems in n dimensions are directly referable to the appropriate theorem in one dimension. Fubini's theorem on the reduction of a double to an iterated integral presents new phenomena. It is handled by noting the fact that it is trivial for step functions, then by appropriate passage to the limit.

Other definitions of the Lebesgue integral are considered. This leads to the proof of the theorems of Egoroff and Lusin.

The final chapter of Part I considers the Lebesgue-Stieltjes integral. First is proved the theorem of Riesz (so fundamental to all modern thinking on abstract integration) to the effect that every bounded linear functional on the space C of functions continuous on $a \leqq x \leqq b$ is of the form $\int_a^b f(x)d\alpha(x)$ where $\alpha(x)$ is of bounded variation—and conversely. This is done by extending the functionals from C to a larger class by precisely the same method used in extending the Lebesgue integral from the class C_0 of step functions to the classes C_1 and C_2 (see above). Questions of the uniqueness of $\alpha(x)$ and concerned with the convergence of functionals are considered. The Lebesgue-Stieltjes integral is sketched briefly from many points of view. Part I closes with a development of the Daniell integral. This is a linear functional defined by extension from a class C_0 of functions $\phi(x)$ defined on an abstract set. The important properties of C_0 are that it be an ordered linear space which is also a lattice (in other words, using the terminology of Bourbaki in his recent book on measure: C_0 est un espace de Riesz). The treatment of this integral throws penetrating light on the Lebesgue and the class of Lebesgue-Stieltjes integrals.

Part II considers first integral equations. This introduces all the phenomena subsequently needed in the study of general linear operators: boundedness, weak, strong, uniform convergence, and the elementary algebraic structure of the ring of operators including inversion and commutativity. If the kernel $K(x, y)$ belongs to $\mathbf{L}^2$ on the square $a \leqq x, y \leqq b$, and gives rise to the operator K on the space L^2 of all functions $h(x)$, $a \leqq x \leqq b$: $Kh(x) = \int_a^b K(x, y)h(y)dy$, then a study is made of the relation between the element $K(x, y) \in \mathbf{L}^2$ and the operator K over L^2. A kernel of the form $\sum_1^n \phi_i(x)\psi_i(y)$ is said to be of finite rank. Any kernel $K(x, y)$ may obviously be approximated in the norm of $\mathbf{L}^2$ by kernels of finite rank. Also, the mapping from kernels into transformations is an isomorphism (into).

Next we find a very thorough discussion of the Fredholm theory. Three methods are given. The first method considers to begin with kernels of finite rank. For these the complete theory is essentially a branch of elementary algebra. To proceed to the general case, it is shown that an arbitrary kernel $K(x, y)$ may be written in the form $K(x, y) = L(x, y) + [K(x, y) - L(x, y)]$ where $L(x, y)$ is of finite rank and the second term $H(x, y)$ is sufficiently small so that $(I-H)^{-1}$

exists. This method is due to E. Schmidt. The second method is the classic one of Fredholm and is sketched rather briefly.

The third method, due to Riesz, is of a geometric nature and applies to any completely continuous operator K. Let $T=I-K$. Let $\mathfrak{M}_n$ be the closed linear manifold of elements f for which $T^nf=0$. Then $(0)=\mathfrak{M}_0\subseteq\mathfrak{M}_1\subseteq\mathfrak{M}_2\subseteq\cdots$. It is shown that there is a first index ν after which equality holds at each step: $\mathfrak{M}_\nu=\mathfrak{M}_{\nu+1}=\cdots$. Now let $\mathfrak{N}_n$ be the range of T^n. Then it may be shown that $\mathfrak{N}_n$ is closed. Clearly $L^2=\mathfrak{N}_0\supseteq\mathfrak{N}_1\supseteq\mathfrak{N}_2\supseteq\cdots$. It may be shown that there is a first index μ after which equality holds at each step: $\mathfrak{N}_\mu=\mathfrak{N}_{\mu+1}=\cdots$. Finally $\nu=\mu$. In the case $\nu=\mu=0$, T has a bounded inverse. If $\nu>0$, $\mathfrak{M}_\nu+\mathfrak{N}_\nu=L^2$ the sum being disjoint, and the pair $(\mathfrak{M}_\nu, \mathfrak{N}_\nu)$ reduces T and hence K. From here to the proof of the Fredholm alternative is a brief step. This Fredholm theory is applied to the solution of Dirichlet and Neumann problems for smooth curves.

The following chapter (V) is devoted to the introduction of abstract Hilbert and Banach spaces. There is the obvious development on linear functionals, adjoint spaces, reflexive spaces, adjoint operators, bases, biorthogonal systems. There is also an interesting section on the form of completely continuous linear transformations in the space C of continuous functions.

The next four chapters are devoted to an exposition of the facts that are associated with symmetric transformations in Hilbert space. Chapter VI treats the completely continuous symmetric transformation. It is shown that if A is such a transformation and if $\{f_n\}$ is a sequence such that $\|f_n\|=1$ and $\|Af_n\|\to\|A\|$ (the norm of A) then there exist solutions to the equation $A\phi=\mu_1\phi$ where $|\mu_1|=\|A\|$. Once such a ϕ has been found, the process of determining the structure of A is based on the application of the same procedure to the subspace of the Hilbert space $\mathfrak{H}$ orthogonal to ϕ. Thus one obtains the sequence of characteristic values $\mu_1, \mu_2, \cdots$ with $|\mu_1|\geqq|\mu_2|\geqq\cdots$ and the associated characteristic elements $\phi_1, \phi_2, \cdots$ which form an orthonormal set. There is also a discussion of the direct determination of the nth characteristic value of A and of the relation existing between the characteristic values of A_1, A_2 and $A=A_1+A_2$. These results are due to Hilbert, Courant, and Weyl. The Hilbert-Schmidt theory of symmetric kernels is treated. (It should be pointed out that a good part of the discussion on completely continuous operators deals with incomplete Hilbert spaces.) It is then applied to two interesting problems. The first one concerns the vibrating string in a formulation given to it recently by Nagy. The second one gives a proof due to Weyl and Rellich of the fundamental theorem of H. Bohr on almost periodic functions to the effect that the generalized Fourier series associated to an almost periodic function $f(x)$ converges in the mean to $f(x)$.

Next (chapter VIII) comes the study of bounded symmetric, normal, and unitary operators. First is a construction and uniqueness proof for the positive square root of a positive symmetric operator. The spectral resolution of an operator A is led to by the necessary discussion of projections and functions of operators. Suppose $u(\lambda)$ is a function on $m \leqq \lambda \leqq M$ where m and M are the bounds of A, $mI \leqq A \leqq MI$; if $u(\lambda)$ may be approximated by a bounded increasing or decreasing sequence of polynomials then the function $u(A)$ may be defined. This process is closely analogous to that for extending the integral from the class C_0 of step functions to the class C_1—see the discussion given earlier. Here we proceed from the class of polynomials to the class of lower or upper semi-continuous functions. This gives the functions E_λ which occur in the resolution of the identity of A and lead to the formula $A = \int_{m-0}^{M} \lambda dE_\lambda$. A second proof of this formula consists in calculating $|A| = \sqrt{A^2}$. The orthogonal manifolds occurring in the projection E_0 are then obviously related to the range and null-manifold of $A^+ = 2^{-1}(A + |A|)$. Unitary and normal transformations are subsequently discussed. There is a section devoted to the Fourier-Plancherel theorem and to Watson transforms.

Now comes a chapter on unbounded transformations. There is first a discussion of the algebra of transformations relative to domains of definition; thus we find a collection of facts concerning $T_1 + T_2$, T_1T_2, T^*; for example $(T_1T_2)^* \subseteq T_2^*T_1^*$. Clearly this type of ground clearing is necessary but it leaves a strange taste in its wake. One feels that this material is being treated in a form which is not final; the definition of the permutability of T with a bounded B is a case in point: $BT \subseteq TB$. As soon as one turns to the graph of a closed transformation one feels much happier. The collection of results on this subject includes the striking theorem that for a closed transformation T, the operators $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$ are bounded and B is positive definite self-adjoint. Examples are given (differential operators) of transformations which are symmetric but not self-adjoint. The first demonstration here given of the integral representation of a self-adjoint transformation based on the above transformation B and C with $T = T^*$ is that of Riesz and Lorch. The second is the original one of von Neumann and transforms the given problem to one on unitary transformations by means of the Cayley transform. The Cayley transform is studied in the case of symmetric non self-adjoint operators. This leads to the notion of the deficiency index. There is given a proof due to Friedrichs of the theorem that a semi-bounded symmetric transformation with lower bound c, $cI \leqq S$, may be extended to a self-adjoint transformation A such that $cI \leqq A$. The more general methods of Krein are discussed in some detail. The problem here is: Let S be symmetric and semi-bounded $cI \leqq S$; let $\gamma \leqq c$. To determine all self-adjoint A such that $A \supseteq S$ and $\gamma I \leqq A$.

Chapter IX begins with a thorough discussion of the calculus based on a self-adjoint transformation A. This calculus assigns a transformation $u(A)$ to a function $u(\lambda)$ measurable with respect to $(E_\lambda f, f) = \|E_\lambda f\|^2$ by means of the formula

$$(u(A)f, g) = \int_{-\infty}^{\infty} u(\lambda) d(E_\lambda f, g).$$

Each such function of A has the property that it permutes with every bounded symmetric transformation which permutes with A,—in symbols $u(A) \smile\smile A$. The converse theorem essentially due to von Neumann is established: If T is a closed transformation for which $T \smile\smile A$, then T is a function $u(A)$ of A providing the Hilbert space is separable. Another theorem of the same author is demonstrated: If $\{X_m\}$ is a denumerable set of permutable self-adjoint transformations there exists a self-adjoint A such that each X_m is a function of A: $X_m = x_m(A)$. From this point forward to the end, the character of the book changes somewhat. Whereas previously all proofs had been given in detail, from now on, the reader is referred to the literature for a complete discussion of some of the stated results.

There is no discussion of spectral multiplicity. We greet its absence like that of a very good friend who is quite demanding and hard to take. The chapter ends with a discussion of the perturbation of the spectrum of a self-adjoint operator. Proof is given of Rellich's theorem: If $A_n \to A$ (strongly) then for the corresponding resolutions of the identity, $E_{n\lambda} \to E_\lambda$ at every point of continuity of E_λ. This device, used in a particular case, led to the first proof of the spectral resolution of A given by Hilbert. For analytic perturbations of A, the following theorem of Rellich and Nagy is proved: Let $A(\epsilon) = A + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \cdots$ be a continuous family of self-adjoint operators. Suppose (μ_1, μ_2) is a real interval having only one point $\lambda^{(0)}$ in the spectrum of $A = A(0)$, and that of finite multiplicity m. Then for ϵ sufficiently small there are $2m$ series

$$\begin{aligned} \lambda_i(\epsilon) &= \lambda^{(0)} + \epsilon\lambda_i^{(1)} + \epsilon^2\lambda_i^{(2)} + \cdots, \\ \phi_i(\epsilon) &= \phi_i^{(0)} + \epsilon\phi_i^{(1)} + \epsilon^2\phi_i^{(2)} + \cdots, \end{aligned} \quad (i = 1, \cdots, m)$$

representing respectively characteristic values and orthogonal characteristic functions of the operator $A(\epsilon)$. That is, the characteristic values and vectors are analytic functions of ϵ. The applications of this theorem to situations in physics are obvious.

The penultimate chapter is devoted to groups and semi-groups of transformations. First comes an elegant proof (due to Nagy) of the theorem of Stone on strongly continuous one-parameter groups of unitary transformations U_t in Hilbert space—the theorem asserting that $U_t = \exp(iAt)$ where A is self-adjoint. A second demonstration based on Bochner's representation of functions of positive type by Stieltjes integrals follows. An immediate application is the mean ergodic theorem of von Neumann. Up to this point, the preceding third of the book has been devoted exclusively to operators in Hilbert space and indeed to normal (including the self-adjoint and unitary) ones: Now the authors branch out to the study of operators over Banach spaces. There are proofs of two theorems of Hille. One asserts the existence on a dense manifold of the infinitesimal generator A of a strongly continuous group T_s of bounded transformations: $A = \lim_{h\to 0} A_h$ where $A_h = h^{-1}(T_h - I)$. Although an exponential representation of T_t is impossible (see unitary case above), the second theorem asserts: $T_t f = \lim_{h\to 0} \exp(tA_h)f$. An intriguing application of this yields the Weierstrass approximation theorem for continuous functions. We turn to ergodic theorems for which the additive semigroup is the set of positive integers. For unitary transformations the elegant proof of Riesz is set forth. The extension of this by Lorch to bounded semi-groups in reflexive spaces: $\|U^k\| \leqq C$, $k = 1, 2, \cdots$, follows. We mention lastly a type of argument originating with G. Birkhoff and applied by F. Riesz to uniformly convex spaces.

The principal section of the last chapter (XI) is devoted to the development of spectral analysis in general Banach spaces by an extension of the Cauchy calculus of residues. Briefly, the extension is carried out as follows: If T is an arbitrary bounded linear transformation, the points z in the complex plane for which $R_z = (T - zI)^{-1}$ exists constitute an open set—the resolvent set. The complementary set, the spectrum $\sigma(T)$ of T, is closed and bounded. Suppose C is simple closed rectifiable curve lying in the resolvent set, then $\int_C R_z dz$ has a meaning in the uniform topology. Furthermore, since R_z is an "analytic function" of z—in accordance with a classical equation: $R_z = R_\zeta + (z-\zeta)R_\zeta^2 + (z-\zeta)^2 R_\zeta^3 + \cdots$—the above integral is unchanged if the curve C is altered slightly. Thus the value of the integral depends only on the portion of the spectrum of T which lies interior to C. It is clear that one may also consider an integral

$\int_C z^n R_z dz$ and that this is equal to $T^n \int_C R_z dz = \int_C T^n R_z dz$ since $(T^n - z^n I)R_z$ is a polynomial in T and z whose integral around C is zero. Consideration of these integrals reveals a remarkable collection of facts concerning the structure of T. The existence of this method was recognized early by Riesz. The times seem to have been so unprepared for it that in all he devoted to it only too brief pages (pp. 117–119) in his book: *Les systèmes d'équations linéaires à une infinité d'inconnues* (1913). For various reasons, this material was soon forgotten. In part, this was due to the fact just cited. In part, events conspired to focus all attention on the symmetric case in Hilbert space for which the present theory was not then needed. It is probable that certain annoying misprints in Riesz' book adjoined to the summary exposition which is there found contributed to the complete disappearance of this theory here presented in germ form. The fact is that in the next quarter century no mention of this material is to be found in the literature of its day.

The interest of the present writer in extending reducibility phenomena from Hilbert space first to reflexive spaces and subsequently to the most general Banach space led to his rediscovering Riesz' earlier work in the form which is presented in the present book. The essential facts are these. For a simple curve C lying in the resolvent set of T, the transformation $P = -(2\pi i)^{-1}\int_C R_z dz$ is a projection, $P^2 = P$, which reduces T, $PT = TP$. Thus the manifolds $\mathfrak{M}$ and $\mathfrak{N}$ determined by P reduce T. The spectrum of T in $\mathfrak{M}$ (where $P(\mathfrak{M}) = \mathfrak{M}$) consists of that part of the spectrum of T lying interior to C. $P = 0$ if and only if the interior of C contains no points of the spectrum. $P = I$ if and only if the entire spectrum of T lies interior to C.

An important and interesting relation exists between the norm $\|T\|$ of T and the spectrum T. Let $|z| \leqq r_T$ be the smallest disc which contains the spectrum $\sigma(T)$ of T. The number r_T is called the spectral radius of T. Then it is easy to see that $\|T\| \geqq r_T$. Since, as follows from an elementary calculation, $\sigma(T^n) = [\sigma(T)]^n$, $\|T^n\| \geqq r_T^n$. Now, in fact, the following theorem holds: $\lim_{n\to\infty} \|T^n\|^{1/n} = r_T$. This theorem was first stated in the present general form by Gelfand in his paper on normed rings (1941). A special case concerned with ring of absolutely convergent trigonometric series had been elaborated by Beurling three years earlier.

This so-called *spectral radius theorem* is a simple consequence of a fundamental identity which appears in the work of this reviewer that deals with the calculus of residues and has already been mentioned above. It is there proved that if T is a linear transformation having no singularities on the unit circle C: $|z| = 1$, then for the projection P

given by the contour integral $P=-(2\pi i)^{-1}\int_C R_z dz$, we have (in the uniform topology)

$$P = \lim_{n\to\infty} (I - T^n)^{-1}.$$

The facts which concern us here may be set down in the form of a

THEOREM. *The following hypotheses concerning T are equivalent*:
(a) *The spectrum of T lies interior to the circle $|z|=1$, that is $r_T<1$.*
(b) $T^n\to 0$ (*uniformly*).
(c) *The $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists, equals r_T, and is less than* 1.

PROOF. Assume (a). If $r_T<1$, then $P=I$ (see a remark made three paragraphs above). Thus $I=\lim_{n\to\infty} (I-T^n)^{-1}$; hence

$$I=\lim_{n\to\infty} (I-T^n) \text{ or } \lim_{n\to\infty} T^n=0.$$

This gives (b).

Assume (b). If $T^n\to 0$, then $\|T^n\|\to 0$ and thus lim sup $\|T^n\|^{1/n}\leqq 1$. Now, clearly, the spectral radius $r_T^{-1}T$ is 1 and for $\epsilon>0$, that of $S=(r_T+\epsilon)^{-1}T$ is less than 1. Thus applying the results just obtained for T to the transformation S, we have lim sup $\|S^n\|^{1/n}\leqq 1$. This implies since ϵ is arbitrary that lim sup $\|T^n\|^{1/n}\leqq r_T$. Since, as mentioned earlier, $\|T^n\|^{1/n}\geqq r_T$, we have lim inf $\|T^n\|^{1/n}\geqq r_T$. Combining these two results yields lim $\|T^n\|^{1/n}=r_T$. This is (c).

Now assume (c). We have immediately, $\|T^n\|\to 0$. Since for the spectra of T and T^n, $[\sigma(T)]^n=\sigma(T^n)$, it follows that $[\sigma(T)]^n\to 0$. Hence the spectrum of T lies in the interior of the circle $|z|=1$. This is (a). This completes the proof of our theorem.

We return to an examination of the present book. The work of Beurling on absolutely convergent series is developed in accordance with his methods. As a corollary one obtains the theorem of Wiener: If a periodic function $f(x)$ has an absolutely convergent Fourier series and does not vanish, then $1/f(x)$ has the same property. There is a section devoted to an operational calculus following the ideas of Riesz, Gelfand, and Dunford. If $u(z)$ is holomorphic on a domain which contains the spectrum of T, and is bounded by a curve C, then the integral $(2\pi i)^{-1}\int_C u(z)R_z dz$ defines the function $u(T)$.

The book terminates with a discussion of the latest work of von Neumann on spectral sets. This theory applies to bounded transformations in Hilbert space. The principal theorem follows: Let C_1 be the disc $|z|\leqq 1$. Suppose $u(z)$ is holomorphic in C_1 and such that $|u(z)|\leqq 1$ in C_1. Then for any T with $\|T\|\leqq 1$, $\|u(T)\|\leqq 1$. A spectral set Z of a transformation T is now defined to be a closed set of the extended complex plane such that for all rational functions $u(z)$ satisfying $|u(z)| \leqq 1$ on Z, the transformation $u(T)$ exists and $\|u(T)\| \leqq 1$. It is proved that unitary and symmetric transformations are characterized by the fact that they have respectively the unit circle and the real axis as spectral sets.

E. R. LORCH

Über die Klassezahl abelscher Zahlkörper. By Helmut Hasse. Berlin, Akademie, 1952.

This is a highly technical book, whose object is the derivation of a formula for the class-number h of an arbitrary absolute abelian field K and the study of this formula. Such a formula had been proved by Kummer for cyclotomic fields (i.e. fields generated by a root of unity) and in the general case by several authors (Fuchs, Beeger, Gut). The source of these formulae is of course the fact that the class number appears in the expression of the residue at $s=1$ of the zeta function $\zeta_K(s)$ of K. Using the product decomposition of ζ_K into L-series, one is reduced to the computation of the values $L(1;\chi)$ at 1 of the L-series corresponding to those characters $\chi \neq 1$ which are associated to K by class field theory. The numbers $L(1;\chi)$ appear as infinite series; the main problem is to express them in closed form, which is done by making use of Gaussian sums.

The resulting formula appears in the form $h=h_0h^*$, where h_0 is the class-number of the maximal real subfield K_0 of K, while h^*, the "second factor" of h, turns out to be an integer >0. The fact that these two factors h_0 and h^* are actually integers is not obvious from the expressions for these numbers which appear in the formula itself. One of the aims of the author is to transform these expressions in such a way as to render their arithmetic nature more apparent. This in itself would not appear so very fascinating a task: when we express the number of zeros of an analytic function in a region by a contour integral, we do not take pains to establish independently that the value of this integral is an integer. However, in the process of so doing, new properties of h_0 and h^* appear which lead to a certain number of new results on class numbers of fields.

The second chapter of the book is concerned with the transformation of the expression for h_0. Here the striving to obtain for h_0 an expression which exhibits it as an integer is not entirely successful. Two different lines of attack are followed which yield results for two different kinds of fields K. The end results of the two methods are in the following form: the product of h by some integer c is expressed as the index in the group of all units of a certain sub-group generated by