theorem. For the most part the development here follows Kodaira and Kakutani and culminates in a proof of their theorem on the completion regularity of Haar measure. Much of this material appears here for the first time in book form. These chapters constitute the focal point of the whole development, a fact which explains to some extent the arrangement and choice of material in the earlier chapters. For instance, properties of Carathéodory outer measure, and most results pertaining specifically to measure in metric spaces, appear only among the exercises. For the same reason, many important results that belong properly to the theory of real functions are omitted altogether. There is no discussion of integration as inverse to differentiation, and the density theorem appears only in a generalized form applicable to Haar measure. The surprising thing, however, is that so much material is included without undue condensation. It seems likely that this book will come to be recognized as one of the few really good text books at its level. It can hardly fail to exert a stimulating influence on the development of measure theory.

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A decision method for elementary algebra and geometry. By Alfred Tarski. Prepared for publication with the assistance of J. C. C. McKinsey. Berkeley and Los Angeles, University of California Press, 1951. 3+63 pp. $2.75.

The results of this monograph were obtained by the author in 1930. The material in its full development was published privately (and hence not reviewed in the Bulletin) by the Rand Corporation in 1948. The present edition is simply a reprint of that edition with corrections and some supplementary notes.

By elementary algebra Tarski means that part of the theory of real numbers which can be expressed in the formal language described as follows: (1) Variables of this language stand for real numbers. (2) There are three constants, "0", "1", and "-1". (3) A term is defined (recursively) as a constant or variable, or else of the form $\alpha+\beta$ or $\alpha\cdot\beta$ where α and β are any terms. (4) An atomic formula is either of the form $\alpha=\beta$ or of the form $\alpha>\beta$ where α and β are terms. (5) A formula is either an atomic formula or made up from atomic formulas by means of truth functions and quantifiers in the usual manner of the first order function calculus. In this language we can discuss any integer and any polynomial we choose; but, although we can talk of all real numbers having a certain property, we cannot talk of *all* integers having a property or of *all* polynomials having a property, since we have no variables to stand for these. In this monograph Tarski presents a decision procedure for elementary algebra, based on Sturm's theorem.

Sturm's theorem itself can be regarded as a decision procedure to determine the truth or falsity of any sentence of the form, "The polynomial p has exactly k roots." Tarski generalizes Sturm's theorem to provide (as a lemma for his main result) a decision procedure for any statement of the following form: "There are exactly k real numbers satisfying σ," where σ is a given system of a finite number of equations in ξ of the form $\alpha_{i,m_i}\xi^{m_i}+ \cdots +\alpha_{i,0}=0$ and inequalities in ξ of the form $\beta_{j,n_j}\xi^{n_j}+ \cdots +\beta_{j,0}>0$. (Cf. Definition 28 and Theorem 29, pp. 31–34.) Then, using this result, Tarski shows that there is a constructive method for determining from any given formula Φ, a formula $U(\Phi)$ which is equivalent to Φ, has no quantifiers, and contains no variables except those appearing free in Φ (Theorem 31, p. 39). From this last theorem the main result follows immediately. For only statements without free variables are true or false, and, for a statement Φ without free variables, $U(\Phi)$ is a truth function of equations and inequalities without variables. It is easy to determine the truth or falsity of such equations and inequalities, whence the truth or falsity of $U(\Phi)$ can be determined by the familiar truth table method.

A formula without free variables is true if it is a true statement in the intuitive sense, concerning the real numbers. Tarski's result then is that there is a decision procedure which determines, for the formal language of elementary algebra, the truth or falsity (in this sense) of every formula with no free variables. Alternatively, this theory could be presented as an axiomatic system; a formula is true in a second sense if it is a theorem of the system. Tarski suggests (in note 9, pp. 48–50) possible axiom systems. The same decision method holds for truth in this second sense, so that the two senses of "truth" are equivalent. It follows from this that such axiom systems are complete and consistent (note 15, pp. 53, 54). Moreover, since these axioms are all satisfied by every real closed field, it follows that any formula of elementary algebra true in one real closed field (as defined e.g. in van der Waerden, Section 70, Chapter IX) is true in every real closed field. Finally, this is still true if we extend elementary algebra by introducing special variables for integers allowing quantification over them. (See supplementary note 7, pp. 62, 63 where the further significance for the theory of real closed fields is explained.)

It may seem paradoxical that the elementary theory of real numbers is decidable (has a decision procedure) while the seemingly simpler elementary theory of natural numbers (i.e. the non-negative integers) is undecidable and in fact incompletable (i.e. has no axiomatic system yielding all and only truths as theorems), as proved by the famous Gödel theorem. The reason for this is probably the fact that the language of elementary natural number theory more fully represents the complexity involved in that theory than the language of elementary algebra represents the complexity of the theory of real numbers. As Tarski points out (in note 13, p. 53), it is impossible to define the notion of a rational number or integer or natural number in elementary algebra; apparently this failure deprives elementary algebra of much of the complexity we ordinarily associate with the theory of real numbers. (Indeed, if the predicate "$NN(x)$" were added to elementary algebra meaning that x is a natural number, then there could no longer be a decision procedure for this enriched language, since the Gödel undecidable statement could be constructed in it.)

In section 3, Tarski gives a language for n-dimensional elementary geometry for a positive integer n, and shows how formulas of this language can be translated into statements of analytic geometry, thereby becoming statements of algebra. Hence elementary geometry is decidable. Certain geometric statements, however, are not regarded as statements of elementary geometry, e.g. statements involving the phrase "all polygons" or the notion of constructibility by means of an arbitrary number of operations of ruler and compass (p. 3).

The Introduction and the many notes are rich with informative and suggestive comments. The Introduction contains a well written, nontechnical summary of the significance of the main result. The proof of the main result should be easy to follow for anyone familiar with symbolic logic, although the reader who has never before seen a proof of Sturm's theorem may wish for a more intuitive account of $G_\xi^k(\alpha,\beta)$ and its rôle. There are a few errata and comments on small points. On page 10 the blank space after line 26 should read "$((\Theta\rightarrow\Phi)\wedge(\Phi\rightarrow\Theta))$." On page 23, line 18, the statement that "division . . . is not available in our system" may be misleading; certainly "x/y" is definable as "the real number z such that $y\cdot z=x$," to which Russell's technique for eliminating descriptions can be applied. On page 23, line 28, insert "and $m\geqq n$" after "β_n of β is 0." (If $m<n$ and $\beta_n=0$ there may be a different kind of undesirable consequence for the notion of a negative remainder.) On page 27, line 13, the word "computed" is badly chosen, since the process indicated in the remainder of the paragraph is not effective.

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