eral recursive functions are indicated: the original method of Herbrand and Gödel, Turing's notion of computability, and Church's notion of $\lambda$-definability. The author mentions Bernays' proof that every function whose values can be calculated in some consistent deductive calculus is general recursive, and describes the normal form of Kleene for general recursive functions. The relationship between general recursive functions and decision methods is pointed out, and reference is made to Church's proof that there is no decision method for the restricted predicate calculus; the more recent extensions of this result by Mostowski, Post, Julia Robinson, R. M. Robinson, and Tarski are not mentioned, however, as they have appeared since Schmidt's article.

The remainder of this part is taken up with an exposition of consistency proofs of various fragments of arithmetic, Gentzen's consistency proof of all of arithmetic using transfinite induction, and a sketch of some unpublished transfinite proofs (due to Lorenzen and Schütte) of the consistency of part of analysis.

Part (C), which is very short, is devoted to a résumé of the theory of types, and of the attempt of Russell and Whitehead (following Frege) to construct arithmetic from logic alone. The author criticizes this attempt on the ground that Russell and Whitehead were unable to prove certain theorems of arithmetic without using axioms which hardly seem to be part of "pure logic" (the axiom of infinity and the axiom of choice).

Part (D) is devoted to an exposition of the intuitionistic mathematics of Brouwer and his followers. The philosophical background of this movement is made clearer here than in the usual presentations.

J. C. C. McKinsey

*Dirichlet's principle, conformal mapping, and minimal surfaces.* By R. Courant. With an appendix by M. Schiffer. (Pure and Applied Mathematics, vol. 3.) New York, Interscience, 1950. 14+330 pp. $4.50.

At the beginning of its history, Dirichlet's principle was one of the occasions for the ascetic enterprise of revising and clarifying the foundations of the calculus of variations. In its latest stage it has proved a powerful tool for the solution of one of the most interesting, difficult, and colorful problems, a solution which could be carried to a generality which, a quarter of a century ago, even the most fanciful optimism would hardly have dared to dream of. No other mathematician is more competent to write a presentation of this subject than is the author of the present book: His first steps in research were still linked to the first part of the history of Dirichlet's principle, and he has been one of the most successful promoters of the development during its latest stage.

Naturally, the scope of the book is limited to Dirichlet's principle in two dimensions. The treatment mainly follows the author's own papers and those of his school, including even the most recent results. As the author stresses in the preface, his foremost aim has not been merely to "present the crystallized product of his thoughts," but he is anxious also to shed light on the development of those thoughts from their very sources. So he does not spare heuristic considerations, motivations of problems, reference to special questions; also many open problems are pointed out. It is needless to say how much this characteristic of the book is to the profit of every reader who is more interested in information than in austere beauty.

In the reviewer's opinion, the geometric aspects of Plateau's and of Douglas's problems come off somewhat badly; it is to be regretted, e.g., that no systematic report on the author's interesting soapfilm experiments has been included.

An appendix, written by M. Schiffer, gives a good picture of the main ideas underlying the recent development of the theory of conformal mapping. Conciseness is gained not by dropping the proofs but by selecting typical cases.

In Chap. I, Dirichlet's principle in its relations to boundary value problems of potential theory is motivated and stated; two proofs are developed in extenso. The simplest applications to conformal mapping are made.

Chap. II deals with an extension which is applicable to conformal mapping of domains of multiple (even of infinite) connectivity onto parallel slit domains. The variational problem and its solution are not restricted to genus zero; in the case of higher genus the result is a representation on a simple parallel slit domain, part of whose slits are ideally coordinated, thus becoming interior lines of a Riemannian domain.

Chap. III is concerned with Plateau's problem in its original form: To span a minimal surface (eventually with the additional condition that it be the surface of least area) into a closed Jordan curve in Euclidean $m$-space, $m \geqq 2$. The solution is based on a variational problem: To minimize the Dirichlet integral $\iint_{u^2+v^2\leqq 1}(\mathfrak{x}_u^2+\mathfrak{x}_v^2)dudv$ within a class of vector functions $\mathfrak{x}(u, v)$, each of which represents an admissible surface. The existence of a solution is proved, and afterwards it is identified as a minimal surface of least area. That it is a minimal surface may be proved, rather quickly, by the use of existence theorems on conformal mapping, or, avoiding such theorems, by consideration of the "first variation of Dirichlet's integral." For the proof of the least area property, theorems on conformal mapping are indispensable. The first variation also yields an independent existence proof, by the method of "safe descent" (an analogue to "steepest descent" for functions of a finite number of variables), whose result, however, cannot be identified as a surface of least area or even as a surface with a relative minimum of area.

Chap. IV is devoted to a generalization of Plateau's problem which was first, and with great success, considered by J. Douglas and is, therefore, here called Douglas's problem: Given any number of Jordan curves in Euclidian $m$-space; to find a minimal surface bounded by these curves with a prescribed Euler characteristic and with a prescribed character of orientability. Again the problem is replaced by a variational problem analogous to the former one; its main new feature is the variability also of the parameter domain within an appropriate class (e.g., all circular rings with outer radius 1 in the case of an orientable surface of genus 0 with two boundary curves). It is rather obvious that one cannot always expect a solution of the problem, since among the limits obtainable from the class of admissible surfaces there are degenerate cases which are not admissible: surfaces with a smaller characteristic or consisting of several parts. So one of the main concerns of this chapter is the discussion of conditions for the existence of a solution.

In the foregoing the solutions of Plateau's and of Douglas's problems are always given by a vector function, harmonic in a certain plane parameter domain, giving a conformal map of the latter on the surface in $m$-space. In the case $m=2$, one finds a conformal representation of the parameter domain onto the domain bounded by the given curves. So proofs are obtained for the existence of a conformal mapping of any multiply-connected domain of genus 0 onto an appropriate member of a certain class of normal domains, admitted as parameter domains. This method is developed in Chap. V. Of course, this time the proof of the "minimal surface character," i.e., of the property that the mapping is conformal, has to be based on the consideration of the first variation of Dirichlet's integral, avoiding all use of existence theorems on conformal mapping. Among the types of normal domains there are very general ones, as yet not accessible to other methods.

The last chapter is devoted to two different problems whose treatment is naturally not as well rounded by far as that of the former ones; many attractive questions are still open. In the first problem, part of the boundary (eventually, with suitable other restrictions, the whole boundary) is left free on some manifold of less than $m$ dimensions. Only special types of this problem are considered. The second question is that of unstable minimal surfaces with prescribed boundary, which has been considered by Morse and Tompkins, and, with another approach, by Shiffman. The short exposition follows the latter author's lines.

Schiffer's appendix deals with such problems as: The relations between Green's function of a multiply-connected domain, Neumann's function, the harmonic measures of the boundary curves, certain special mapping functions, kernel functions, etc.; certain extremal problems; the variation of Green's function considered as a function of the domain, with applications.

Two bibliographies, one for the main part and one for the appendix, are inserted.

It may well be hoped that this lucid presentation of results obtained up to now will prove a sound basis and a stimulus for further research.

H. GRUNSKY

*Mathematische Grundlagen der höheren Geodäsie und Kartographie.* Vol. I. *Das Erdsphäroid und seine konformen Abbildungen.* By R. König and K. H. Weise. Berlin, Springer, 1951. 18+522 pp. 49.60 DM.

The mathematical theory of cartography and related problems in higher geodesy has provided frequent inspiration for basic studies in geometry, function theory, and conformal transformation. Most mathematicians have been inclined however to dismiss the subject as a fairly elementary exercise in analysis and have left its problems to the geographers and map makers. A small group has nevertheless maintained a continuing interest in this field which was set on a sound mathematical basis by the early work of Lambert, Lagrange, and Gauss.

The present work represents a significant contribution to the few outstanding works in this field. It stands along with the three major references: M. A. Tissot, *Mêmoire sur la représentation des surfaces géographiques et des projections des cartes*, Paris, 1881; L. Driencourt and J. Laborde, *Traité des projections des cartes géographiques*, Paris, 1932; and W. Jordan and O. Eggert, *Handbuch der Vermessungskunde*, 8th ed., Stuttgart, 1939–1941. These three together with the present treatment provide a comprehensive background and guide for research and application.