BOOK REVIEWS

The kernel function and conformal mapping. By Stefan Bergman. (Mathematical Surveys, vol. 5.) American Mathematical Society, New York, 1950. 8+160 pp. $4.00.

Orthogonal functions and the kernel function have become fundamental tools of the modern theory of functions; they allow one to consider the single-valued regular functions which are of integrable square over a given region as the elements of a Hilbert space.

The starting point for this development was L. Bieberbach's minimum problem connected with the Riemann mapping problem (see Rend. Circ. Mat. Palermo vol. 38 (1914)). The first researches on orthogonal functions in the complex domain originated under the influence of the Berlin school (particularly Erhard Schmidt and L. Bieberbach). Stefan Bergman, himself a descendent of this school, has intensively developed the theory of orthogonal functions and their kernels with great thoroughness during many years. He has now produced a text which provides an introduction to this now extensive field.

Chapter 1 begins with the following definition of the function class $\mathfrak{L}^2$ on which the whole theory rests: $f(z)$ is called of class $\mathfrak{L}^2$ in the given schlicht bounded region B if (1) $f(z)$ is regular and single-valued and

$$(2) \qquad \mathfrak{F}_B(f) = \iint_B |f(z)|^2 d\omega < \infty.$$

The orthonormal systems $\phi_\nu(z)$ are introduced in $\mathfrak{L}^2$, the Riesz-Fischer theorem is proved, and it is shown that there are closed orthonormal systems associated with B.

$$K(z, \bar{t}) = \sum_{\nu=1}^{\infty} \phi_\nu(z)\overline{\phi_\nu(t)}$$

is called the kernel. It is defined everywhere in B and is independent of the choice of the $\phi_\nu(z)$.

In Chapter 2 it is first shown that the uniquely determined function $f(z)$ in B, with $f(t)=1$, $t \in B$, and $\iint_B |f(z)|^2 d\omega$ a minimum, is $f(z) = K(z, \bar{t})/K(t, \bar{t})$. Then some essentially more general minimum problems are solved.

Chapter 3 introduces an invariant metric with the aid of the kernel function. This is the Bergman metric, which is a true Riemannian metric and indeed a Kähler metric. It has turned out to be superior in many ways to the metric of H. A. Schwarz, Pick, and Carathéodory.

In Chapter 4 the author considers arbitrary classes of functions $f=\sum c_\nu f_\nu$, $\sum |c_\nu|^2<\infty$, where the f_ν form an orthonormal system. (The functions of these classes therefore represent a Hilbert space.) If then the kernel function

$$K(P, Q) = \sum_{\nu=1}^{\infty} f_\nu(P)\cdot\overline{f_\nu(Q)}$$

exists for arbitrary points P, Q of the fundamental region B, most of the results of the first three chapters can be extended to these more general classes of functions.

There is evidently a close relationship between statements about analytic functions and about sets of points in Hilbert spaces. This is developed for Vitali's theorem in a posthumous paper of Otto Toeplitz which appeared too late to be taken account of in this book (see Comment. Math. Helv. vol. 23 (1949) pp. 222–242).

Chapter 5 contains principally the important relationship among the Green's function, the Neumann function, and the kernel function for the class of harmonic functions,

$$k(z, \zeta) = (2\pi)^{-1}[N(z, \zeta) - G(z, \zeta)].$$

The next chapter uses these functions to establish the mappings of all p-ply (p finite) connected regions on the various types of canonical slit regions.

Chapter 7 takes up another orthogonalization (by integrals along the boundary). This yields a new kernel function $\hat{K}_B(z, \bar{\zeta})$. In simply connected regions

$$\left\{\frac{\hat{K}_B(z, \bar{\zeta})}{\hat{K}_B(\zeta, \bar{\zeta})}\right\}^2 = \frac{K_B(z, \bar{\zeta})}{K_B(\zeta, \bar{\zeta})}.$$

The close connection between this kernel function $\hat{K}$ and the class of bounded functions is studied. If $f(z)$ is regular and single-valued in B and $|f(z)| \leqq 1$ there, then, for every $\zeta\in B$, $|f'(\zeta)| \leqq 2\pi\hat{K}_B(\zeta, \bar{\zeta})$.

Chapter 8 is concerned with the dependence of the functions $K(z, \zeta)$, $G(z, \zeta)$, and $N(z, \zeta)$ on the region B. The variations of these functions are calculated when B is replaced by a somewhat larger region. Under certain assumptions,

$$\delta K(z, \bar{\zeta}) = \int_b K(z, \bar{t})K(t, \bar{\zeta})\delta n\cdot ds + o(\epsilon)$$

if the boundary b is displaced by an amount $\delta n = \epsilon\rho(s)$ in the direction of the inner normal. Later the more general variational method of M. Schiffer and the numerous results obtained by means of it are discussed.

While in the preceding chapters the existence of the Green's function, the Neumann function, etc., is assumed (which is permissible since the proofs can be found in many places in the literature), this borrowing is now dispensed with and the slit mapping is carried through without this assumption.

In Chapter 10 it is shown that the methods applied in the preceding chapters can also be applied to the solutions of partial differential equations of elliptic type. Here there are unexpected results from recent investigations of Bergman and of Bergman and Schiffer. The chapter ends with a treatment of the equation of elasticity, $\Delta\Delta\phi = 0$, in order to show how the process must be modified for equations of higher order.

The final chapter is concerned with functions of two complex variables and the analytic (pseudo-conformal) mappings generated by them. It is written for readers who are already familiar with the foundations of the theory of functions of several complex variables. First special regions (bicylinder, hypersphere, etc.) are treated; then the orthogonal functions are introduced for arbitrary schlicht bounded domains, and the mappings on representative regions by means of minimal functions and the invariant metric are set up. Finally the author discusses regions with distinguished boundary surfaces, the corresponding Bergman integral representations, and the "extended classes of functions." The choice of the topics in this chapter is perhaps somewhat too much oriented in the direction of the author's own extensive publications.

However, the book as a whole gives a distinguished introduction to the theory of orthogonal functions with its abundance of new results.

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Théorie des distributions. By Laurent Schwartz. (Publications de l'Institut de Mathématique de l'Université de Strasbourg, nos. 9 and 10; Actualités Scientifiques et Industrielles, nos. 1091 and 1122.) Vol. I, 1950, 148 pp. Vol. II, 1951, 169 pp.

In Euclidean E_k we consider a general function $\varphi(x) = \varphi(x_1, \cdots, x_k)$ which is defined and infinitely differentiable everywhere and is zero outside a bounded domain $D = D_\varphi$, and, as in a previous context, we call such a function a *testing function.* Next, if $F = F(x)$ is a fixed