THE DECEMBER MEETING IN PASADENA

The four hundred seventy-sixth meeting of the American Mathematical Society was held at the California Institute of Technology, Pasadena, California, on Saturday, December 1, 1951. Approximately 140 persons attended, including the following 89 members of the Society:

O. W. Albert, H. L. Alder, T. M. Apostol, Richard Arens, M. M. Beenken, Clifford Bell, M. C. Bergen, Benjamin Bernholtz, H. F. Bohnenblust, J. V. Breakwell, R. E. Bruce, Herbert Busemann, W. D. Cairns, Paul Civin, P. H. Daus, Robert Davies, A. C. Davis, E. A. Davis, R. Y. Dean, D. B. Dekker, A. H. Diamond, R. P. Dilworth, Milton Drandell, Roy Dubisch, D. G. Duncan, W. D. Duthie, H. A. Dye, Arthur Erdélyi, Harley Flanders, G. E. Forsythe, O. A. Gross, G. J. Haltiner, H. J. Hamilton, V. C. Harris, A. R. Harvey, L. A. Henkin, A. D. Hestenes, M. R. Hestenes, P. G. Hodge, P. G. Hoel, Alfred Horn, D. H. Hyers, J. R. Jackson, C. G. Jaeger, P. B. Johnson, D. H. Lehmer, M. M. Lemme, J. W. Lindsay, Y. L. Luke, J. L. McGregor, J. C. C. McKinsey, M. W. Maxfield, C. B. Morrey, T. S. Motzkin, J. W. Odle, Barrett O'Neill, R. S. Pierce, D. H. Potts, W. T. Puckett, Edgar Reich, J. B. Robinson, R. M. Robinson, P. G. Rooney, Herman Rubin, I. J. Schoenberg, Raymond Sedney, Seymour Sherman, G. E. F. Sherwood, I. S. Sokolnikoff, R. H. Sorgenfrey, M. L. Stein, Robert Steinberg, E. G. Straus, A. C. Sugar, J. D. Swift, Alfred Tarski, A. E. Taylor, F. B. Thompson, D. L. Thomsen, F. A. Valentine, R. L. Vaught, Morgan Ward, W. R. Wasow, M. A. Weber, J. G. Wendel, P. A. White, A. L. Whiteman, František Wolf, F. H. Young.

By invitation of the Committee tò Select Hour Speakers for Far Western Sectional Meetings, Professor J. C. C. McKinsey of Stanford University delivered at 11:00 A.M. an address entitled *Notions and problems of game theory*. Professor McKinsey was introduced by Professor H. F. Bohnenblust. There were sessions for contributed papers in the morning and afternoon, presided over by Professors A. E. Taylor, D. H. Lehmer, and C. B. Morrey. Following the meetings, those attending were guests at tea in the Athenaeum.

Following are abstracts of papers presented at the meeting. Papers with abstract numbers followed by "*t*" were presented by title. Paper number 96 was presented by Professor Hodge and number 85 by Mr. Thompson. Professor Kalicki was introduced by Professor Alfred Tarski, and Dr. Selmer by Professor T. M. Apostol.

ALGEBRA AND THEORY OF NUMBERS

75. T. M. Apostol: *Theorems on generalized Dedekind sums.*

The sums in question are defined by $s_p(h, k) = \sum_{\mu=1}^{k-1} (\mu/k)\overline{B}_p(h\mu/k)$, where $\overline{B}_p(x)$ is the pth Bernoulli function and h and k are relatively prime integers. For odd $p>1$, the author derives the formula (1): $s_p(h, k) = ip!(2\pi ik)^{-p}\sum_{\mu=1}^{k-1} \cot(\pi h\mu/k)\zeta(p, \mu/k)$, where $\zeta(s, a)$ is the Hurwitz zeta function, defined for $R(s)>1$ by the series

$\sum_{n=0}^{\infty} (n+a)^{-s}$. Formula (1) is then applied to give an analytic proof of the reciprocity law for $s_p(h, k)$. If p is allowed to tend to 1 in (1), the second member tends to $-(2\pi k)^{-1}\sum_{\mu=1}^{k-1} \cot (\pi h\mu/k)\Gamma'(\mu/k)/\Gamma(\mu/k)$, which is shown to be equivalent to an expression obtained by Rademacher for the Dedekind sums, namely: $s_1(h, k) = (4k)^{-1}\sum_{\mu=1}^{k-1} \cot (\pi h\mu/k) \cot (\pi\mu/k)$. The paper also includes an elementary derivation of this last equation using finite Fourier series. (Received October 22, 1951.)

76*t*. H. W. Becker: *Combinatory interpretations of the differences of the numbers of E. T. Bell.*

Apply the Steffensen operation $\nabla U_n = U_n - U_{n-1}$ to the elementary Bell numbers $@_n = e^{\Delta}O^n$. The first interpretations of $\nabla^m @_n$ were in terms of *forbidden* positions in rhyme schemes, distributions, and non-attacking rook patterns on a triangular chess board (Mathematical Magazine vol. 22 (1948) p. 25). Two new interpretations are: $\nabla^{n-p+1}@_n$ is the number of $n+1$ letter rhyme schemes whose last singleton, or else last "a," is in the pth position (except that the enumeration for one "a" is $@_n$, not $\nabla^n @_n$) with isomorphic distribution and rook meanings. These afford a combinatory proof that $\sum_0^n \nabla^m @_n = \nabla^{n+2}@_{n+2} = @'_{n+1}$, the number of $n+1$ letter rhyme schemes not without singletons. The iterated Bell numbers (Ann. of Math. vol. 39 (1938) p. 539; Amer. J. Math. vol. 61 (1939) p. 89) have analogous breakdowns, in terms of more complicated difference operators. (Received October 17, 1951.)

77. Anne C. Davis: *Cancellation theorems for products of order types*. I. Preliminary report.

Let $\alpha, \beta, \cdots, \kappa, \cdots$ denote arbitrary order types. The (ordinal) sum and the (ordinal) product of α, β are respectively denoted by $\alpha+\beta$ and $\alpha\cdot\beta$. A type κ is here called a *cancelling* type if $\kappa\cdot\alpha = \kappa\cdot\beta \rightarrow \alpha = \beta$ for arbitrary types α, β. Theorem I. *The following three conditions are equivalent*: (i) κ *is a cancelling type*. (ii) $\kappa \neq \kappa\cdot\alpha$ *for every type* $\alpha \neq 1$. (iii) $\kappa \neq \kappa + \kappa\cdot\alpha + \kappa$ *for every type* α. Theorem II. *If* κ *is a scattered type* (*i.e. the type of a scattered order set*) *and* $\kappa \neq 0$, *then* $\kappa \neq \kappa+\mu+\kappa$ *for every* μ. A consequence of Theorems I and II is Corollary III. *Every nonzero scattered type is a cancelling type.* Corollary III is an improvement of the result, published without proof by Lindenbaum (Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie vol. 19 (1926) p. 321) that every nonzero ordinal is a cancelling type. Another consequence of Theorem I is Corollary IV. *Every type* κ *of a nonempty ordered set without gaps and without first and last elements is a cancelling type.* Hence, in particular, the type λ of the set of real numbers is a cancelling type (a fact previously noted by Tarski). (Received October 15, 1951.)

78. D. G. Duncan: *A formula in Littlewood's algebra of S-functions.*

Let $t_i = \{\mu\} \otimes S_i$ where $\otimes$ denotes Littlewood's "new" multiplication of S-functions. In this note the generating function $\prod_{i=1}^{\infty} e^{(t_i/i)z^i} = \sum_{r=0}^{\infty} (\{\mu\} \otimes \{r\})z^r$ is used to establish the identity: $\{\mu\} \otimes \{2r\} = \sum_{j=1}^{r-1} (\{\mu\} \otimes \{2r-j\})(\{\mu\} \otimes \{j\})(-1)^{j+1} + 2^{-1}(\{\mu\} \otimes \{r\})^2(-1)^{r+1} + 2^{-1}\sum_{\alpha} (1/\alpha_1! \cdots \alpha_n!)(t_2/1)^{\alpha_1} \cdots (t_{2r}/r)^{\alpha_r}$. An application of this formula arises in invariant theory where a systematic study of $\{\mu\} \otimes \{r\}$ is desired. (Received October 23, 1951.)

79. Harley Flanders: *Generalization of a theorem of Ankeny and Rogers.*

The following result is proved: Let k be an algebraic number field and ν the largest integer such that $\cos (2\pi/2^\nu)$ is in k. Let a be a nonzero element of k, n a natural number, and assume that the congruence $x^n \equiv a \pmod{\mathfrak{p}}$ is solvable for almost all prime divisors $\mathfrak{p}$ of k. Then either a is a perfect nth power, $a = b^n$, or $a = b^n(2^{-1}(1+\cos (2\pi/2^\nu)))^{n/2}$ with b in k. The converse is true. Also a short proof of the Ankeny-Rogers result (k = rationals) is given, based on an extension of the Eisenstein irreducibility lemma. Finally a corresponding theorem for function fields is given. (Received October 29, 1951.)

80. Alfred Horn: *The normal completion of a subset of a complete lattice and lattices of continuous functions.*

Let C be a subset of a complete lattice B. A construction is given for a subset of B which is isomorphic with the normal completion $\tilde{C}$ of C. As an application, let B be the set of all real-valued (including $\pm\infty$) functions on a topologic space X. If C is the class of continuous members of B, then $\tilde{C}$ becomes the class of normal lower semicontinuous functions, provided that X is completely regular. Also treated are the cases where C is the class of finite-valued or the class of bounded continuous functions. The latter case was first discussed by R. P. Dilworth (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 427–438). New characterizations of normal lower semicontinuous functions are obtained. These are applied to prove that for any two topologic spaces, the lattices of normal lower semicontinuous functions are isomorphic if and only if the lattices of regular open sets are isomorphic. The sufficiency was first proved in the completely regular case by Dilworth (loc. cit.). (Received October 17, 1951.)

81. Jan Kalicki: *On equational equivalence of abstract algebras.*

Terminology used will be that of Birkhoff (*On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. vol. 31 (1935)). Two abstract algebras are equationally equivalent if their sets of laws coincide. A decision method is obtained for equational equivalence of finite abstract algebras. It is proved that if an equation $\phi = \psi$ is a law of a finite algebra A_1 of order m_1 without being a law of a finite algebra A_2 of order m_2, or vice versa, then there is an equation $\phi' = \psi'$ with the same property and with the number of different variables not exceeding $m = m_1 m_2$. Also if there is an equation with the properties of $\phi' = \psi'$, then there is an equation $\phi'' = \psi''$ with the same properties and such that the ranks of ϕ'' and ψ'' do not exceed the number $\sum_{i=1}^{m} m^{m^i}$. Since for given A_1 and A_2 the family of equations with the properties of $\phi'' = \psi''$ is determined and finite, the decision method follows. By applying the method to A_1 and to direct product $A_1 \times A_2$ it can be found whether the set of laws of A_1 is included in that of A_2. Similar results are obtained for logical matrices. (Received October 18, 1951.)

82*t*. Peter Scherk: *On sets of integers.* II.

(1) The results of the preceding abstract are extended from sets of non-negative integers to arbitrary sets $A, B, C, \cdots$ of integers. (2) The following corollary is noted: Let $A + B \subset C$, $r \subset A$, $s \subset B$. Let $h \geqq 0$, $0 < \gamma < 1$, and $A(r, r+k) + B(s, s+k) \geqq \gamma k$ $[k = 0, 1, \cdots, h]$. Then $C(r+s, r+s+h) \geqq \gamma h$. Here $D(x, y) = \sum_{x < d \leqq y,\, d \subset D} 1$. (Received October 18, 1951.)

83. E. S. Selmer: *A conjecture of Mordell concerning rational points on cubic surfaces.* Preliminary report.

It is well known that the elementary congruence conditions (e.c.c.)—together with solubility in real numbers—are sufficient for the solubility in integers of a homogeneous quadratic equation in any number of variables. The author has recently (Acta Math. vol. 85 (1951)) shown the insufficiency of the e.c.c. for a homogenous *cubic* equation in *three* variables. Prof. L. J. Mordell (*Rational points on cubic surfaces*, Publ. Math. Debrecen vol. 1 (1949) p. 1) has conjectured that the e.c.c. are sufficient for solubility of a homogeneous cubic equation in *four* variables, i.e. for the existence of rational points on cubic surfaces. The author considers the purely cubic equation $a_1x_1^3+a_2x_2^3+a_3x_3^3+a_4x_4^3=0$, with the additional condition that (for instance) a_1a_2/a_3a_4 is a rational cube. In this case the equation can be written as $x^3+my^3=n(z^3+mu^3)$. By factorizing both sides in the purely cubic field $R(m^{1/3})$, the author proves Mordell's conjecture for this particular equation. To cover all cases, a deep result of H. Hasse is needed. The same method does not apply to the general purely cubic equation, but it seems very likely that Mordell's conjecture still holds. (Received November 29, 1951.)

84. E. G. Straus: *A problem of D. H. Lehmer.*

Lehmer conjectured that every number theoretical function $f(x)$ with the property that it maps every complete residue system modulo any prime, except for a finite number of exceptional primes, onto a complete residue system modulo this prime, is of the form $f(x)=ax+b$, where a is divisible only by exceptional primes and b is an arbitrary integer. This conjecture is proved and certain generalizations are established. (Received October 19, 1951.)

85. Alfred Tarski and F. B. Thompson: *Some general properties of cylindric algebras.* Preliminary report.

Given any ordinal α, let $\mathfrak{A}$ be a Boolean algebra $\mathfrak{B}=\langle A, +, \cdot, ^-, 0, 1\rangle$ with additional systems of unary operations C_ξ and distinguished elements $d_{\xi,\eta}$ satisfying (for $x, y\in A$ and $\xi, \eta, \zeta<\alpha$) the postulates: P_1. $C_\xi 0=0$; P_2. $x\cdot C_\xi x=x$; P_3. $C_\xi(x\cdot C_\xi y)=C_\xi x\cdot C_\xi y$; P_4. $C_\xi C_\eta x=C_\eta C_\xi x$; P_5. $d_{\xi,\xi}=1$; P_6. $\xi, \eta\neq\zeta\rightarrow d_{\xi,\eta}=C_\zeta(d_{\xi,\zeta}\cdot d_{\eta,\zeta})$; P_7. $\xi\neq\eta\rightarrow C_\xi(d_{\xi,\eta}\cdot x)\cdot C_\xi(d_{\xi,\eta}\cdot\bar{x})=0$. $\mathfrak{A}$ is called an *α-dimensional cylindric algebra with diagonal elements*—a CA_α; for examples see following abstract. An ideal in $\mathfrak{A}$ is a set X which is a Boolean-algebraic ideal in $\mathfrak{B}$ such that $C_\xi x\in X$ whenever $x\in X$, $\xi<\alpha$. Connections between ideals and homomorphisms in CA_α's are the same as in Boolean algebras. $\mathfrak{A}$ proves to be simple if and only if, for every $x\neq 0$, there is a finite sequence $\xi_0, \cdots, \xi_{n-1}<\alpha$ with $C_{\xi_0}\cdots C_{\xi_{n-1}}x=1$. For $x\in A$ let the *dimension index* $Dx=\text{E}_\xi(\xi<\alpha,\ C_\xi x\neq x)$. Direct factors in $\mathfrak{A}$ prove to be sets of the form $\text{E}_x\ (x\leqq a)$ where Da is empty. $\mathfrak{A}$ is directly indecomposable if and only if Dx is nonempty for every $x\neq 0, 1$. A CA_α in which Dx is finite for every x is called *locally finitely-dimensional*—an FCA_α. Simple FCA_α's coincide with directly indecomposable FCA_α's. Hence, by a theorem of Birkhoff, every FCA_α is a subdirect product of simple FCA_α's. (Received October 16, 1951.)

86. Alfred Tarski: *A representation theorem for cylindric algebras.* Preliminary report.

For notations see preceding abstract. X being any set, X^α is the set of all sequences of length α with all terms in X. A *proper* CA_α is a system $\mathfrak{A}$ in which: 1 is a union of some pairwise disjoint sets Y_i^α; 0 is the empty set; $+$ is set-addition, $\cdot$ set-multiplication, $^-$ complementation (relative to 1); $C_\xi X$ (for $X\subseteq 1$, $\xi<\alpha$) is the set of all sequences $y\in 1$ such that $y_\eta=x_\eta$ for some $x\in X$ and for every $\eta<\alpha$, $\eta\neq\xi$; $d_{\xi,\eta}$ ($\xi, \eta<\alpha$) is the set of all $x\in 1$ with $x_\xi=x_\eta$; A is any field of sets $X\subseteq 1$ closed under the operations C_ξ and containing the sets $d_{\xi,\eta}$. Clearly, P_1–P_7 hold in every proper CA_α. By carrying over Gödel's completeness theorem for predicate calculus to CA_α's the following result is obtained: *For $\alpha \geqq \omega$, every algebraic equation which holds (identically) in every proper FCA_α holds also in every FCA_α (and in fact in every CA_α).* The following theorem improving this result has been established: *Every FCA_α with $\alpha \geqq \omega$ is isomorphic to a proper FCA_α.* The problem is open whether this theorem extends to arbitrary CA_α with $\alpha \geqq \omega$. For $\alpha < \omega$ some results can be derived from what is available in the literature. (Received October 16, 1951.)

87t. R. L. Vaught: *On the equivalence of the Axiom of Choice and a maximal principle.*

As a particular case of Zorn's Lemma, one obtains the following statement: (1) Every family of sets has a maximal disjointed subfamily, i.e., a maximal subfamily having the property that any distinct two of its members are disjoint. Thus (1) is a consequence of the Axiom of Choice. The question was asked by Tarski whether conversely (1) implies the Axiom of Choice. The answer is affirmative. In fact, suppose $\mathcal{A}$ is any disjointed family of nonempty sets. Let $\mathcal{B}$ be the family of all sets of the form $\{A, \{x\}\}$ with $x \in A \in \mathcal{A}$. By (1), $\mathcal{B}$ has a maximal disjointed subfamily $\mathfrak{M}$. Let M be the set of all x such that for some A, $\{A, \{x\}\} \in \mathfrak{M}$. From the maximality of $\mathfrak{M}$ it follows easily that M is a set having exactly one element in common with each member of $\mathcal{A}$. (Received October 19, 1951.)

Analysis

88t. L. D. Berkovitz: *On double trigonometric integrals.*

Let ϕ be a complex-valued measure which is finite-valued on all bounded Borel sets in the plane and whose total variation over a circle with center at (u, v) and radius $1/2$ is $o((u^2+v^2)^{-1/2})$. The first principal result of this paper is that there exists a trigonometric series $T = \sum_{m,n=-\infty}^{\infty} a_{mn} e^{i(mx+ny)}$ with coefficients $a_{mn} = o((m^2+n^2)^{-1/2})$ such that T is circularly equiconvergent with the integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(xu+yv)} d\phi(u, v)$. The second result is the analogous theorem stated for restricted equiconvergence, the conditions on the variations of ϕ and the order of the a_{mn} now being $o((|u|+1)^{-1/2} \cdot (|v|+1)^{-1/2})$ and $o((|m|+1)^{-1/2}(|n|+1)^{-1/2})$ respectively. To prove the two theorems the concept of formal product for integrals is defined and multiplication theorems analogous to those used in the Rajchman-Zygmund approach to localization for single trigonometric series are developed. Riemann formulae are then established and from these the principal results follow. (Received October 15, 1951.)

89. Paul Civin: *Multiplicative closure and the Walsh functions.*

A study is made of real orthogonal systems which are closed under multiplication and which contain a unit. Any such system which is infinite is shown to be a multiplicative group isomorphic with the group of the Walsh functions. Furthermore, there is a measurable transformation of the unit interval onto itself which transforms the Walsh functions into the given system of functions and such that the inverse image of a set of measure α is of measure α. (Received September 27, 1951.)

90t. Abolghassem Ghaffari: *Topological nature of hodograph transformation.*

When considering the flow around an aerofoil, one observes that there is usually a stagnation point at $z=0$. Let $\phi + i\psi = Az + Bz^2 + \cdots$ represent the complex potential near the stagnation point. Since the velocity is zero at $z=0$, then $A=0$ and therefore

$d(\phi+i\psi)/dz=u-iv=2Bz+\cdots$, where $u-iv=w\exp(-i\theta)=w_m\zeta^{1/2}\exp(-i\theta)$ in which $\zeta=w^2/w_m^2$ and θ denote hodograph variables; u and v indicate the components of fluid velocity w; w_m is the maximum of w. It is found that $\phi+i\psi=C(u-iv)^2+\cdots$, where $C=1/4B$. Therefore $\phi+i\psi$ can be expanded in a power series in $\zeta^{1/2}$ without constant term and no term in $\zeta^{1/2}$; the first term involves ζ. At greater distance from the aerofoil the complex potential may be expanded in the form $\phi+i\psi=Az+B\log B+C/z+\cdots$, where A is the complex velocity in the main stream and B depends on the circulation. It is shown that $\phi+i\psi\sim AB/(u-iv-A)+B\log B/(u-iv-A)+P(u-iv-A)$, where $P(u-iv-A)$ is a power series in $(u-iv-A)$, and hence the singularities in the hodograph plane at $u-iv=A$ are a combination of single poles and logarithmic singularities. Therefore no finite set of elementary solutions can represent the true state of the field of flow around an aerofoil. (Received October 11, 1951.)

91*t*. Joseph Kampé de Fériet: *Uniqueness theorem for the heat equation on an infinite rod.*

If $f(x)\in L^2$ and $u(x, t)$ satisfies in the half-plane $t>0$ the following conditions: (1)u_x and u_{xx} exist, u_{xx} being everywhere finite, (2) u and $u_{xx}\in L^2$ with respect to x for every t, (3) $u_{xx}(x, t)=\text{l.i.m.}\ [u(x, t+h)-u(x, t)]:h$ if $h\to 0$, (4) $f(x)=\text{l.i.m}\ u(x, t)$ if $t\to +0$, then: (5) u_t exists everywhere, (6) $u_t=u_{xx}$, (7) $u(x, t)$ is equal to the Poisson integral: $(4\pi t)^{-1/2}\int_{-\infty}^{+\infty}\exp[-(s-x)^2/4t]f(s)ds$. This proposition is an improvement of the uniqueness theorem given by S. Bochner, K. Chandrasekharan, *Fourier transforms*, p. 134. The proof is based on the following facts: (a) due to (1), (2): $u_x\in L^2$ and $(u, u_{xx})=-\|u_x\|^2$; (b) due to (3): the derivative of $\|u(x, t)-f(x)\|^2$ is equal to $2(u, u_t)\leqq 0$. (Received October 29, 1951.)

92*t*. M. S. Klamkin: *On series derived from* Ln$(1+x)$ *by successive integrations.*

By evaluating the p-fold integrals $\int_0^x\cdots\int_0^x(dx)^p/(1+x)$ and $\int_0^x\cdots\int_0^x(dx)^p/(1-x)$ we can determine the sums of series whose nth terms are given by $X^{p+n}n!/(p+n)!$, $(-1)^{n+1}X^{p+n}n!/(p+n)!$, $X^{p+2n}2n!/(p+2n)!$, and $X^{p+2n+1}(2n+1)!/(p+2n+1)!$. The integrals are evaluated by expressing each as a polynomial in x plus a logarithmic term. The coefficients of the polynomial are determined from a difference equation of the type $a_{r,p}=a_{r,p-1}/(p-r)+\phi(r, p)$ subject to the condition $a_{p,p}=0$ and where $\phi(r, p)$ is known. (Received October 18, 1951.)

93. Y. L. Luke: *An associated Bessel function.*

The purpose of this paper is to investigate the properties of the solution to $\Delta_v y=z^2d^2y/dz^2+zdy/dz-(v^2+z^2)y=e^{-z}z^{\mu+1}$. The technique follows the pattern for the development of the well known Lommel functions. The result is applied to evaluate explicitly some indefinite integrals involving Bessel functions of the type previously considered by the author (*Some notes on integrals involving Bessel functions*, Journal of Mathematics and Physics vol. 29 (1951) pp. 27–30). The analysis is extended to the evaluation of some infinite integrals of interest. Other applications are also given. (Received September 28, 1951.)

94*t*. František Wolf: *Rellich's theorem on analytic perturbation for operators in Banach space.*

Rellich's theorem (Math. Ann. vol. 113 (1936) pp. 600–619) is concerned with an operator $A(\lambda)$ in Hilbert space which is self-adjoint for real λ and depends analytically on the parameter λ. If $A(0)$ has an isolated spectral value μ_0 of finite multiplicity m, then the theorem asserts that there exist m analytic functions $\mu_k(\lambda)$, $k=1, \cdots, m$, $\mu_k(0)=\mu_0$, which for sufficiently small λ represent all spectral values of $A(\lambda)$ near μ_0. This theorem has been generalized by S. L. Jamison (Thesis, Berkeley, 1949) to operators in Hilbert space which are normal for real λ's. The author generalized this theorem (to appear in Math. Ann.) to operators in Banach space which have the following property. There exists a sequence $\{\lambda_n\}$, $\lim \lambda_n=0$, such that the Banach algebra generated by $A(\lambda_n)$ has no radical. (Received October 19, 1951.)

Applied Mathematics

95t. Abolghassem Ghaffari: *On a solution of the hodograph equations in the supersonic region.* Preliminary report.

Let ψ and ϕ denote the stream function and the velocity potential of a compressible supersonic flow and let ζ, $1/6<\zeta<1/2$, and θ be the hodograph variables. The simplified form of the hodograph equations in the supersonic range is (1) $P\phi_\zeta=\psi_\theta$, $Q\psi_\zeta=\phi_\theta$, where $P=2\zeta(1-\zeta)^{\beta+1}[(2\beta+1)\zeta-1]^{-1}$, $Q=2\zeta(1-\zeta)^\beta$, and $\beta=1/(\gamma-1)$; γ is adiabatic index. The author gives a solution of (1) in the supersonic region. Considering the new variable t defined by $t=\int_{\zeta_s}(PQ)^{-1/2}d\zeta$ and setting $R=(Q/P)^{1/2}$, where $\zeta_s=1/6$ is the value of ζ at the sonic speed, the equations (1) become (2) $(R\psi_t)_t=R\psi_{\theta\theta}$, $(R^{-1}\phi_t)_t=R^{-1}\phi_{\theta\theta}$. It is found that for the approximate value $R\sim t^{1/3}$, the equations (2) transform into (3) $\psi_{tt}+1/3t\psi_t=\psi_{\theta\theta}$, $\phi_{tt}-1/3t\phi_t=\phi_{\theta\theta}$ which are satisfied by the solutions (4) $\psi=-t^{1/3}\sum A_mJ_{1/3}(mt)$, $\phi=t^{2/3}\sum B_mJ_{2/3}(mt)$, where the J's denote Bessel functions and A_m, B_m are arbitrary constants. (Received October 11, 1951.)

96. P. G. Hodge and William Prager: *Limit design of reinforcements of cut-outs in slabs.* Preliminary report.

A thin square slab with a central circular cut-out is subjected to a slowly increasing uniformly distributed tensile stress on the four edges. The flow limit is determined, i.e., that value of the tensile stress for which unrestricted plastic flow sets in. Reinforcing rings welded to the two sides of the slab are then designed so that the reinforced slab has the same flow limit as an unreinforced slab without cut-out. The analysis is based on the assumptions of plane stress and Tresca's yield criterion of constant maximum shearing stress. (The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received October 15, 1951.)

97. František Wolf: *On Mann's singular integral equation.*

W. R. Mann and the author discussed (Quarterly of Applied Mathematics vol. 9 (1951) pp. 163–184) the integral equation $y(t)=\int_0^t\mathcal{G}(y(\tau))/(\pi(t-\tau))^{1/2}\cdot d\tau$ which arose from a problem in the conduction of heat with a nonlinear boundary condition. $\mathcal{G}$ is a nonincreasing continuous function in (0, 1) with $\mathcal{G}(1)=0$. While the discussion arrived at satisfactory results in the case of $\mathcal{G}$ satisfying a Lipschitz condition, in the case of the more general conditions on $\mathcal{G}$ given above only existence of a solution was proved. The uniqueness of the solution seemed to lie rather deep. Later J. H. Roberts and W. R. Mann proved the last two results in a paper to be published in the Pacific Journal of Mathematics. They discuss the integral equation $y(t)=\int_0^tK(t-\tau)\mathcal{G}(y(\tau))d\tau$ with certain conditions on the kernel K. They attack the problem directly and while uniqueness proved easy, the increasing character of the solution required a long and delicate proof. The author was able to prove the same results by approximating the general problem by special problems where conditions imposed upon $\mathcal{G}$ and K make the results easily accessible. The proofs prove to be simple, using standard methods, and are so general that their ideas may very well apply to many similar problems. (Received October 19, 1951.)

Geometry

98. D. B. Dekker: *The curves of Darboux and a classification of surfaces.*

On a surface in three-dimensional Euclidean space the differential equation for the Darboux curves is obtained in tensor form. The coefficients in the differential equation are found to be the triply covariant components of a third order tensor relative to transformations of the surface coordinates. This tensor is expressed in terms of the first and second fundamental second order surface tensors and the covariant derivative of the second fundamental tensor. It is then possible to study the Darboux curves on a surface without transforming to a particular type of surface coordinate system. Also a simple method of classifying surfaces by means of their first and second fundamental tensors is obtained by using the above third order tensor. (Received October 16, 1951.)

99. Milton Drandell: *Generalized convex sets in the plane.*

Let $\{C\}$ be a family of curves in the complex plane satisfying: (1) each member of $\{C\}$ is a closed Jordan curve passing through the point at infinity; (2) through any two finite points there passes a unique member of $\{C\}$. It is proved that if $\{p_n\}$ and $\{q_n\}$ converge to distinct finite points p_0 and q_0 respectively, and if $C_n \in \{C\}$ passes through p_n and q_n, $C_0 \in \{C\}$ passes through p_0 and q_0, then $\{C_n\}$ converges to C_0 in much the same sense as uniform convergence for real continuous single-valued functions. If $C \in \{C\}$ and p is a finite point not on C, then through p there passes a member of $\{C\}$ having no finite point in common with C. It is shown that this member is not unique. A set E is defined to be convex relative to $\{C\}$ in a manner analogous to the usual notion of convexity. If E is closed, bounded, possesses interior points, and is convex relative to $\{C\}$, then (a) E is the closure of a region; (b) the boundary of E is a closed Jordan curve. If F is compact, then its convex hull is compact. Several differences between this concept and ordinary convexity are discussed. (Received October 5, 1951.)

100t. Peter Scherk: *Convex bodies off center.*

Let $0<t<1$; $n \geqq 2$. Given a distance function $F(\mathfrak{p})$ in n-space. Suppose $F(\mathfrak{p}) \leqq 1$ defines a centrally symmetric convex body of volume J and suppose $\mathfrak{p}=0$ is the only integral vector satisfying $F(\mathfrak{p}) \leqq 2tJ^{-1/n}$. Then to every vector $\mathfrak{a}$ there is an integral vector $\mathfrak{g}$ such that $F(\mathfrak{a}-\mathfrak{g}) < (1+(n-1)t^n)/t^{n-1}J^{1/n}$. This improves results by Mahler (Neder. Akad. Wetensch. vol. 51 (1938) pp. 634–637) and Störmer and Walter (Archiv der Mathematik vol. 2 (1950) pp. 346–348). (Received October 18, 1951.)

Topology

101t. R. C. Blanchfield: *Self-linking invariants of homotopy chains.* Preliminary report.

Let G be a finitely generated additive abelian group, with the ring $\mathfrak{o}$ of a number field F as operator domain, which is paired to itself by a linking function V (see

Burger, Math. Zeit. vol. 52 (1949) pp. 217–255). Calculable invariants for the equivalence class of V, analogous to those defined for the rational case in a recent paper of Fox and Blanchfield (Ann. of Math. vol. 53 (1951) pp. 556–564), are defined. First the domain of operators is extended to a suitable principal ideal ring $\mathfrak{o}' \subset F$. The torsion coefficients of the group G with operators $\mathfrak{o}'$ are elements $\tau_1, \cdots, \tau_n$ of $\mathfrak{o}'$; say $\tau_{i+1} | \tau_i$ and let $\tau_{n+1} = 1$. For each r, $1 \leqq r \leqq n$, there exists an r-tuple $(a_1, \cdots, a_r)$ of elements of G for which the element $D(a_1, \cdots, a_r) = \tau_1 \cdots \tau_r \cdot \det \| V(a_i, a_j) \|$ of $\mathfrak{o}'$ is $\not\equiv 0 \pmod{\tau_r/\tau_{r+1}}$. If $(a_1, \cdots, a_r)$ and $(b_1, \cdots, b_r)$ are two such r-tuples, then $D(a_1, \cdots, a_r)$ and $D(b_1, \cdots, b_r)$ are equivalent in the sense that there exists an element $\xi \in \mathfrak{o}'$ such that $D(a_1, \cdots, a_r) \equiv \xi \bar{\xi} D(b_1, \cdots, b_r) \pmod{\tau_r/\tau_{r+1}}$. These equivalence classes are analogous to the quadratic residue classes and reduce to them in the rational case. Linking functions of the type considered arise in the study of $(4n+3)$-dimensional manifolds (see Burger, loc. cit.); the above mentioned equivalence classes are topological invariants of the manifold. (Received October 5, 1951.)

102. R. P. Dilworth: *The space of normal upper semicontinuous functions.*

Let $N(S)$ denote the set of normal upper semicontinuous functions on a completely regular space S. It has been shown (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 427–438) that $N(S)$ is lattice isomorphic with the set of real continuous functions on the Boolean space $\mathfrak{B}$ associated with the Boolean algebra of regular open sets of S. Following Čech, the functions of $N(S)$ may be mapped into continuous functions on a compact Hausdorff space as follows: Let I_f denote the closed real interval $(\inf_S f, \sup_S f)$. Let $\mathfrak{S}$ denote the closure in the Cartesian product $\prod_f I_f$ of the points $\prod_f f(x)$ where $x \in S$. The correspondence $f \rightarrow F$ where $F(t) = t_f$ maps $N(S)$ into continuous functions on $\mathfrak{S}$. Now let $\mathfrak{S}$ be partially ordered by the relation $x \supseteq y$ if and only if $f(x) \geqq f(y)$ all $f \in N(S)$. The following results are obtained: (i) $\mathfrak{S}$, as a partially ordered set, has minimal elements, (ii) the minimal elements form a closed subset $\mathfrak{C}$ of $\mathfrak{S}$, (iii) $\mathfrak{C}$ is homeomorphic to $\mathfrak{B}$. (Received October 16, 1951.)

103*t*. S. T. Hu: *On homotopy groups of homogeneous spaces.*

Let G be a simply-connected topological group. If G is locally arcwise connected and locally simply-connected and Z is a discrete normal subgroup of G, then it is well known that the fundamental group of the topological quotient group $X = G/Z$ is isomorphic with the discrete group Z. Since, in this case, G is a covering space of X, it follows from a theorem of Hurewicz that $\pi_n(X) \approx \pi_n(G)$ for the higher homotopy groups. The purpose of this paper is to remove the inessential conditions imposed upon G and Z in these classical theorems. Indeed, the following generalization is the main theorem of the paper. If G is a simply-connected topological group and Z is either a zero-dimensional closed normal subgroup of G or a zero-dimensional locally compact closed subgroup of G, then the fundamental group of the homogeneous space $X = G/Z$ of the left cosets of Z in G is isomorphic with the abstract group Z and $\pi_n(X) \approx \pi_n(G)$ for the higher homotopy groups. (Received October 16, 1951.)

104. J. R. Jackson: *On homotopy groups of function spaces.*

Let X be an absolute neighborhood retract, X_0 be a retract of X, Y be an arbitrary topological space, and $y_0 \in Y_0 \subset Y$. Let Ω be the space of continuous functions on X into Y carrying X_0 into Y_0, Ω_0 be the space of continuous functions on X into Y

carrying X_0 to y_0, and Φ be the space of continuous functions on X_0 into Y_0. Each function space is given the compact-open topology. Each function space contains a function carrying all of the domain into y_0, and these functions we denote by y_0. Then the mth homotopy group $\Pi_m(\Omega, y_0)$ of Ω at base point y_0 is a split extension of a subgroup isomorphic to $\Pi_m(\Omega_0, y_0)$ by $\Pi_m(\Phi, y_0)$. In particular, if $m \geqq 2$, then $\Pi_m(\Omega, y_0)$ is isomorphic to the direct sum of $\Pi_m(\Omega_0, y_0)$ and $\Pi_m(\Phi, y_0)$. These results contain most of the known theorems on homotopy groups of function spaces, and also make it possible to determine the structures—in terms of the homotopy groups of Y and Y_0—of many homotopy groups inaccessible to previous results. (Received October 22, 1951.)

105. R. S. Pierce: *The Boolean algebra of regular open sets.*

Let S be a completely regular topological space. Denote by $C(S)$ the lattice of bounded, real-valued, continuous functions on S. A partial ordering of $C(S)$ is defined as follows: $f \supseteq g$ if, for all $h \in C(S)$, $f \wedge h \leqq 0$ implies $g \wedge h \leqq 0$. Writing $f \sim g$ if $f \supseteq g$ and $g \supseteq f$ defines a congruence relation on $C(S)$. Let L be the lattice obtained from $C(S)$ by identifying equivalent elements. The main result of the paper is that L is isomorphic to a sublattice of B, the complete Boolean algebra of regular open sets of S, and, moreover, that B is isomorphic to the normal completion of L. (Received October 19, 1951.)

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