THE OCTOBER MEETING IN WASHINGTON

The four hundred seventy-third meeting of the American Mathematical Society was held at the National Bureau of Standards on Saturday, October 27, 1951, in conjunction with a meeting of the Institute of Mathematical Statistics. This meeting was a portion of the celebration of the fiftieth anniversary of the founding of the National Bureau of Standards. The registered attendance at the meeting was 225 and included the following 180 members of the Society.

J. C. Abbott, Milton Abramowitz, M. I. Aissen, T. W. Anderson, R. P. Bailey, J. H. Barrett, L. K. Barrett, E. F. Beckenbach, E. G. Begle, E. E. Betz, Gertrude Blanch, R. C. Blanchfield, J. H. Blau, Joseph Blum, H. W. Bode, H. F. Bohnenblust, J. R. Bowman, A. T. Brauer, R. S. Burington, L. J. Burton, G. H. Butcher, F. P. Callahan, W. R. Callahan, H. C. Carter, W. C. Carter, E. A. Coddington, L. W. Cohen, Natalie Coplan, E. F. Cox, G. F. Cramer, A. B. Cunningham, J. H. Curtiss, G. B. Dantzig, D. A. Darling, C. R. DePrima, J. B. Diaz, W. F. Donoghue, William H. Durfee, Samuel Eilenberg, Benjamin Epstein, J. L. Ericksen, Trevor Evans, F. D. Faulkner, N. J. Fine, F. H. Fowler, L. K. Frazer, F. N. Frenkiel, L. M. Fulton, A. S. Galbraith, Landis Gephart, H. H. Germond, H. E. Goheen, Michael Goldberg, Leon Goldstein, R. A. Good, R. D. Gordon, W. H. Gottschalk, E. C. Gras, J. W. Green, R. E. Greenwood, D. W. Hall, Marshall Hall, Jr., J. F. Hannan, L. S. Hart, Philip Hartman, H. J. Hasenfus, E. V. Haynsworth, G. A. Hedlund, Vaclav Hlavatý, A. J. Hoffman, Temple Hollcroft, L. A. Hostinsky, E. A. Hoy, D. R. Hughes, Witold Hurewicz, S. B. Jackson, F. E. Johnston, Joseph Kampé de Fériet, R. E. Keirstead, M. S. Klamkin, S. H. Lachenbruch, Jack Laderman, O. E. Lancaster, Lamar Layton, Patrick Leehey, Marguerite Lehr, B. A. Lengyel, J. H. Levin, D. C. Lewis, T. P. G. Liverman, B. J. Lockhart, D. B. Lowdenslager, Eugene Lukacs, E. J. McShane, Irwin Mann, Murray Mannos, C. G. Maple, W. H. Marlow, M. H. Martin, W. T. Martin, Joseph Milkman, C. E. Miller, D. D. Miller, H. J. Miser, Don Mittleman, R. W. Moller, E. W. Montroll, C. N. Moore, T. W. Moore, W. E. Moore, C. R. Morris, W. R. Murray, P. P. Nesbeda, I. L. Novak, M. W. Oliphant, Alex Orden, J. C. Oxtoby, L. E. Payne, Anna Pell-Wheeler, I. D. Peters, G. W. Petrie, J. W. Ponds, F. M. Pulliam, O. J. Ramler, C. J. Rees, M. S. Rees, J. N. Rice, R. P. Rich, P. R. Rider, H. P. Robertson, L. V. Robinson, L. R. Sario, S. S. Saslaw, I. R. Savage, J. B. Scarborough, A. T. Schafer, R. D. Schafer, Henry Scheffé, G. E. Schweigert, I. E. Segal, W. H. Sellers, J. A. Silva, R. C. Simpson, A. D. Solem, J. J. Sopka, G. L. Spencer, I. A. Stegun, W. J. Strange, E. G. Swafford, Olga Taussky, J. H. Taylor, B. J. Tepping, Feodor Theilheimer, W. R. Thickstun, J. A. Tierney, John Todd, C. B. Tompkins, A. W. Tucker, J. W. Tukey, S. M. Ulam, J. L. Vanderslice, A. H. Van Tuyl, M. C. Waddell, D. H. Wagner, G. C. Webber, H. F. Weinberger, Harry Weingarten, Alexander Weinstein, M. E. White, P. M. Whitman, G. T. Whyburn, L. S. Whyburn, R. F. Williams, N. Z. Wolfsohn, M A. Woodbury, D. M. Young, F. H. Young, E. H. Zarantonello, Neal Zierler, J. A. Zilber.

Professor J. C. Oxtoby, of Bryn Mawr College, addressed the Society at 2:00 P.M. on *Ergodic sets* by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. This session was presided over by Dr. S. M. Ulam.

There were sessions for the presentation of contributed papers at 10:00 A.M. and 3:15 P.M. presided over by Professor H. P. Robertson and Professor C. N. Moore respectively.

At 1:30 P.M. members of the Society and of the Institute were welcomed by Dr. A. V. Astin, Acting Director of the National Bureau of Standards, at a joint session presided over by Dr. J. H. Curtiss, Chief of the Applied Mathematics Division.

Abstracts of the papers presented at the meeting are listed below, those with the letter "*t*" after their numbers having been read by title. Papers numbered 9 and 27 were read by Dr. Taussky and Dr. Carter respectively. Mr. Boone was introduced by Professor K. L. Chung, Mr. Leslie and Dr. Love by Professor L. C. Young.

ALGEBRA AND THEORY OF NUMBERS

1*t*. A. A. Aucoin: *Systems of Diophantine equations.*

Solutions of the following Diophantine systems are given: (1) $\prod_{j=1}^{n} \sum_{k=1}^{q} a_{ijk}x_k = f_i(y_i)$ $(i=1, \cdots, n)$, where $f_i(y_i) = f_i(y_{i1}, \cdots, y_{iq})$ are homogeneous polynomials of degree m with integral coefficients and m and n are relatively prime. (2) $f_1(x) = g_1(u)$, $f_2(x) = g_2(v)$, where $f_1(x) = f_1(x_1, \cdots, x_p)$, $f_2(x) = f_2(x_1, \cdots, x_q)$ are homogeneous polynomials of degree n and are such that integers $x_i = a_i$ exist for which all the partials of f_1 as well as those of f_2, of all orders less than $n-1$, vanish. g_1 and g_2 are homogeneous polynomials with integral coefficients of degree m where m and n are relatively prime. (3) $f_1(x_i, y_i, z_i) = g_1(x_i, y_i, z_i)$, $f_2(x_i, y_i, z_i) = g_2(x_i, y_i, z_i)$, where the functions involved are polynomials, homogeneous in each of the sets of variables x_i, y_i, z_i. The solution depends upon the degree of homogeneity. The method applies to more than two equations. (Received August 15, 1951.)

2*t*. H. W. Becker: *Planar rhyme schemes.*

Among forty odd other interpretations, $M_n = (2n, n)/(n+1)$ is the number of planar rhyme schemes of n letters. (Nonplanar rhyme schemes, such as *abab*, have crossovers in their Puttenham diagrams.) A lexicon theorem ranks them individually. Known and new breakdowns of M_n have planar rhyme significations: ${}_{\text{I}}M_{n,m} = M_{n-m} \cdot M_{m-1}$ (Euler), last "*a*" in mth position; ${}_{\text{II}}M_{n,m} = (2n-m-1, n-m)m/n$ (reverse Delannoy), m a's, initial ascending run of length m, $m+1$ progeny in M_{n+1}; ${}_{\text{III}}M_{n,m} = 2^{n-1-2m}(n-1, 2m)M_m$ (Touchard), m inversions; ${}_{\text{IV}}M_{n,m} = (n, m-1)(n-1, m-1)/m$ (W. H. Wise), m different letters, m ascending runs, $m-1$ rising pairs; ${}_{\text{V}}M_{n,m} = (n, m)\Delta^m M_0 = (n, m)N_m$, m singletons (unrhymed letters); ${}_{\text{VI}}M_{n,m} = (n-1, m)\Delta^m M_1$, m couplets (consecutive-rhyme pairs); ${}_{\text{VII}}M_{n,m} = N^{n-m}(N+1)^{m-1}$, last singleton in mth position. These lead to variations of the classic generating function for M_n. Thus $N_n = (M-1)^n$, the subset of M_n without singletons, has the generating function $1/(1-tN) = [1 - \{(1-3t)/(1+t)\}^{1/2}]/2t$. (Received September 12, 1951.)

3*t*. H. W. Becker: *Network severances.*

Given a net $m \times n$ subject to simultaneous failure of one knot in each row, what are the possibilities ${}_mS_n$ of severances of the net? The difference equations through $m=6$ are set up, solution of the last one of the second degree being ${}_4S_n=2F_{2n}$ (F the Fibonacci series). A more interesting approach is ${}_mS_n=m\cdot 3^{n-1}-2V_n$, the subtrahend being an edge effect correction. $V_n=3V_{n-1}+W_{n-1}$, where W_n is the number of rhyme schemes of n letters, such that consecutive letters are of the form k, k or k, $k\pm 1$. In turn, $W_n=3W_{n-1}-X_{n-1}$, where $X_n=\Delta^n M_1={}^aW_n$, the membership of W_n terminating in "a". There are numerous relations between V_n, W_n, X_n, and $M_n=(2n, n)/(n+1)$. For example, if ${}^m\Delta U_n=U_{n+1}-m\cdot U_n$ (J. Touchard, International Mathematical Congress, Toronto, pp. 465–472), the case $m=3$ inverts to $W_{n+1}=(3-M)^n$, $V_{n+1}=M^{-1}(3-M)^n(-)^{n-1}$, and the elegant dual $X_{n+1}=M(3-M)^n$, $M_{n+1}=X(3-X)^n$. Typical of their generating functions is $1/(1-tX)=[3t-1-\{1-t(2+3t)\}^{1/2}]/2t$. There are also numerous subset isomorphisms with planar rhyme schemes. (Received September 12, 1951.)

4. W. W. Boone: *An extension of a result of Post.* II.

Magnus (J. Reine Angew. Math. vol. 103) terms a certain alteration in the form of a word of a group the *elimination of a generator*. The problem of determining when—for a given word—a similar elimination, described below, can be carried out is unsolvable. One can explicitly exhibit: a particular group, $G_{T'}$, with a *finite* number of defining relations R_i on a *finite* number of generators, $g_1, g_2, \cdots, g_n$, such that it is recursively unsolvable to determine for an arbitrary positive word W of $G_{T'}$ (i.e., a word in which all exponents are positive) whether or not W is expressible as a positive word W' not containing g_1 and made up only of generators occurring in W. Alternatively, regarding exponentiation as an abbreviation, words as made up of the distinct *symbols*, g_i^{+1}, g_i^{-1}, and $g_i^{+1}g_i^{-1}=g_i^{-1}g_i^{+1}=$ THE VOID WORD as among the R_i, the foregoing obviously may be formulated as an (unsolvable) problem regarding the *elimination of a symbol*. The result follows by a comparison of proofs of (cf. abstract, *An extension of a result of Post*, Journal of Symbolic Logic vol. 16, no. 3) $\sum\vdash_6 W_1$ and $\sum Bq_0B\Omega\vdash_6 W_2$, T_6 amended to represent $G_{T'}$, B a free generator, W_1 and W_2 arbitrary words, Ω such that every T_5-symbol not a q-symbol occurs therein. (Received September 13, 1951.)

5. A. T. Brauer: *Limits for the characteristic roots of a matrix.* V.

It was shown in part 2 of this paper (Duke Math. J. vol. 14 (1947) pp. 21–26) that each characteristic root of an arbitrary square matrix of order n must lie in the interior or on the boundary of at least one of certain $n(n-1)/2$ ovals of Cassini which are determined by the elements of the matrix. It is shown in this paper that this result remains correct if each of these ovals is replaced by another oval which in general is smaller. From this result an improvement of the well known theorem of Minkowski on positive determinants is obtained. (Received September 13, 1951.)

6t. Jesse Douglas: *On finite groups with two independent generators.*

Although the theory of finite groups is a well worked branch of mathematics, the study of the present topic in its full generality seems not to have been previously undertaken. A detailed account will appear in a series of notes to be published in Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951). (Received July 12, 1951.)

7t. Jesse Douglas: *On the invariants of finite abelian groups.*

This paper offers a new proof, of a simple and illuminating nature, of the classic theorem which characterizes any finite abelian group to within isomorphism by its invariants—certain prime-power factors of its order. Some new interpretations of these invariants are brought to light in terms of functions $I(x)$, $J(x)$, $K(x)$ determined by the structure of the group—e.g., $p^{I(x)}$ is the number of elements whose period is a divisor of p^x. (The group is presumed to be of prime-power order p^α, which entails no essential loss of generality.) Graphical considerations are adduced: e.g., the graph of $I(x)$ is always rising and concave downward (both in the wide sense), the slope of its initial segment is equal to the number of elements in any basis, elements of order p^x are present in a basis if and only if the graph has a corner at the point x, and the number of such elements is equal to the "curvature" of the graph at that point, etc. The detailed account will appear in Proc. Nat. Acad. Sci. U.S.A. vol. 37 (1951). (Received July 11, 1951.)

8. Marshall Hall, Jr.: *A combinatorial problem on abelian groups.*

If $\binom{a_1, \cdots, a_n}{c_1, \cdots, c_n}$ is a permutation of the elements of a finite abelian group A, then the sum $b_1 + \cdots + b_n$ of the differences $c_i - a_i = b_i$, $i = 1, \cdots, n$, is zero (the group operation being addition). It is shown conversely that if we are given elements b_i with $b_1 + b_2 + \cdots + b_n = 0$, then there exists a permutation with these b's as its differences in some order. (Received August 20, 1951.)

9. T. S. Motzkin and Olga Taussky: *Matrices with property* L.

Two $n \times n$ matrices A and B are said to have property P—considered by Frobenius, McCoy and other—(or property L, considered by Kac) if there exists an arrangement $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_n$ of their characteristic roots such that any function (or linear function) $f(A, B)$ has $f(\lambda_i, \mu_i)$ as its characteristic roots. Pairs of matrices with property L are investigated, and it is shown in particular that hermitian matrices with property L have property P, are commutative, and can be transformed into diagonal matrices by the same similarity. Matrices with property L and $n > 2$ do not in general have property P. (Received August 23, 1951.)

10. D. H. Wagner: *On free products of groups.* Preliminary report.

It is proved that the homomorphisms on a free group F onto a free product are characterized as those homomorphisms (onto) which map some free basis of F into the union of the free factors. This result was, in effect, obtained by I. Gruschko (Rec. Math. (Mat. Sbornik) N. S. vol. 8 (1940) pp. 169–182) for the case where F is finitely generated. The present proof uses a transfinite convergence procedure developed by H. Federer and B. Jónsson (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 1–27) in conjunction with a procedure developed by the writer for reducing a finite subset of a free product. These reductions are motivated by J. Nielsen's reductions for free groups (Matematisk Tidsskrift (1921) pp. 77–94) and possess similar properties. An example is given of a free product which is a homomorphic image of a group which cannot be decomposed into a free product. (Received April 28, 1951.)

Analysis

11*t*. A. A. Aucoin: *A generalization of Abel's transformation.*

Abel's transformation is generalized to $\sum_{j=n+1}^{n+k} a_j^p b_j = \sum_{j=n+1}^{n+k} A_j^p (b_j - b_{j+1}) - A_n^p b_{n+1} + A_{n+k}^p b_{n+k+1} - \sum_{i=0}^{p-2} C_{p,i+1} \sum_{j=n}^{n+k-1} A_j^{p-1-i} a_{j+1}^{i+1} b_{j+1}$, where $A_k = \sum_{r=0}^{k} a_r$ and p is an integer > 1. Among the theorems on infinite series proved with this transformation are two which follow when $b_j \equiv 1$. The first is: If $\sum_{j=0}^{\infty} a_j^p$ converges to S and if $\sum_{j=0}^{\infty} A_j^{p-1-i} a_{j+1}^{i+1}$ converge to S_i $(i = 0, \cdots, p-2)$, then A_n^p converges to $S+$

$\sum_{i=0}^{p-2} C_{p,i+1} S_i$. The second is a corresponding theorem when the a's are functions of a variable x. (Received August 15, 1951.)

12*t*. E. A. Coddington: *A characterization of ordinary self-adjoint differential systems.*

If D denotes the set of all complex-valued functions of class C^n, $n \geqq 1$, on a bounded interval $a \leqq t \leqq b$, the formal linear differential operator L of order n is defined for all $x \in D$ by $L(x) = p_0 x^{(n)} + p_1 x^{(n-1)} + \cdots + p_n x$, where the p_m are complex-valued functions of class C^{n-m}, and $|p_0(t)| \neq 0$, on $a \leqq t \leqq b$. Suppose L coincides with its Lagrange adjoint L^*, defined for all $x \in D$ by $L^*(x) = (-1)^n(\bar{p}_0 x)^{(n)} + (-1)^{n-1}(\bar{p}_1 x)^{(n-1)} + \cdots + \bar{p}_n x$. Let $l_i(x) = \sum_{j=1}^{n} M_{ij} x^{(j-1)}(a) + \sum_{j=1}^{n} N_{ij} x^{(j-1)}(b) = 0$, $i = 1, \cdots, n$, be n linearly independent boundary conditions with complex constants M_{ij}, N_{ij}. It is shown that the differential system consisting of L and the conditions $l_i(x) = 0$, $i = 1, \cdots, n$, is self-adjoint if and only if $MB^{-1}(a)M^* = NB^{-1}(b)N^*$, where M and N are the matrices with elements M_{ij}, N_{ij} respectively, and $B(t)$ is the nonsingular skew-hermitian matrix (depending only on the coefficients p_m in L) associated with the semi-bilinear form in Green's formula. This result can be used to characterize self-adjoint operators in the Hilbert space $L^2(a, b)$ which arise in a natural way from the formal operator L. (Received September 11, 1951.)

13*t*. E. A. Coddington and Norman Levinson: *A boundary value problem for a nonlinear differential equation with a small parameter.*

The purpose of this paper is to relate the existence, uniqueness, and general behavior of the solution $y = y(x, \epsilon)$, for small $\epsilon > 0$, of the two-point boundary value problem (B): $\epsilon y'' + f(x, y)y' + g(x, y) = 0$, $(' = d/dx)$, $y(0) = y_0$, $y(1) = y_1$, with the solution $u = u(x)$ of the corresponding "degenerate" initial value problem (I): $f(x, u)u' + g(x, u) = 0$, $u(1) = y_1$. Theorem 1 (Existence). Let $(0, y_0)$, $(1, y_1)$ be two points in the real (x, y)-plane, and assume: (1) f, g are real functions such that the problem (I) has a solution $u = u(x)$ on $0 \leqq x \leqq 1$, with $u(0) \geqq y_0$; (2) f, g have continuous first derivatives in a region $R: 0 \leqq x \leqq 1$, $|y - u(x)| \leqq a$, $a > 0$, which includes the point $(0, y_0)$; (3) there exists a constant $k > 0$ such that $f > k$ for (x, y) in R. Then, for all sufficiently small $\epsilon > 0$, there exists in R a solution $y = y(x, \epsilon)$ of (B). Further, $y(x, \epsilon) \to u(x)$, $y'(x, \epsilon) \to u'(x)$, as $\epsilon \to 0$, uniformly on any subinterval $0 < \delta \leqq x \leqq 1$. Theorem 2 (Uniqueness). Under the assumptions of Theorem 1, for sufficiently small $\epsilon > 0$, there exists at most one solution $y = y(x, \epsilon)$ of (B) in any region $R_0: 0 \leqq x \leqq 1$, $|y - u(x)| \leqq a_0 < a$, $a_0 > 0$. The theorems hold under slightly weaker, but more complicated, assumptions. If $\epsilon < 0$, the role of the left and right boundaries $x = 0$ and $x = 1$ are interchanged. (Received August 15, 1951.)

14. D. A. Darling: *On the distribution of the roots of certain almost periodic functions.*

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be rationally independent real numbers and $f(t) = \sum_{i=1}^{n} a_i \sin \alpha_i t$. Let the positive roots of $f(t)$ be $t_1 < t_2 < \cdots < t_i < \cdots$ and define $N_k(x)$ as the number of those intervals (t_{i+1}, t_i) whose length is $\leqq x$ for $i = 1, 2, \cdots, k$. Then $\lim_{k \to \infty} N_k(x)/k = F(x)$ exists and it is a matter of some theoretical and practical interest to find this distribution function. The following result is proven: Let $X_1, X_2, \cdots, X_n$ be independent random variables, each having a uniform distribution over $(0, 2\pi)$; let ξ^* be the smallest positive and ξ_* be the largest negative root of the equation $\sum a_i \sin (X_i + \alpha_i \xi) = 0$, and let $G(x)$ be the distribution function of the random variable $U=(\xi^*-\xi_*)(\sum\alpha_i^2)^{1/2}$. Then $F(x)=(E(U^{-1}))^{-1}\int_0^x(1/t)dG(t)$. Under certain restrictions on the $\{a_i\}$ and $\{\alpha_i\}$, in the n-dimensional space the $(n-1)$-dimensional surface $\sum_{i=1}^n a_i \sin x_i=0$ and a system of lines having direction numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are considered. Then $F(x)$ gives the distribution of lengths of the secants intercepted by two adjacent sheets of the surface. The mean and variance have a correspondingly simple geometrical interpretation. In general, if $p_1(t), p_2(t), \cdots, p_n(t)$ are arbitrary functions having a common period, the roots of $\sum a_i p_i(\alpha_i t)$ can be treated in the above manner. (Received September 4, 1951.)

15*t*. H. Margaret Elliott: *On approximation to analytic functions by rational functions.*

Let C consist of a finite number of mutually exterior analytic Jordan curves. Let $w=\phi(z)$ map the exterior of C conformally onto $|w|>1$ so that the points at infinity in the two planes correspond; denote by C_R the locus $|\phi(z)|=R>1$. Let $f(z)$ be analytic in the interior of C and continuous on the corresponding closed region $\overline{C}$. Suppose $f^{(k)}(z)$ exists and is continuous on C with modulus of continuity $\omega(\delta)$, $\omega(\delta)\neq 0$. Let points α_{nj}, $j=1, \cdots, n$; $n=1, 2, \cdots$, be given on or exterior to C_A, $A>1$. It is proved that there exist rational functions $r_n(z)=(a_{n0}z^n+a_{n1}z^{n-1}+\cdots+a_{nn})/(z-\alpha_{n1})\cdots(z-\alpha_{nn})$ such that $|f(z)-r_n(z)|\leqq M\omega(1/n)/n^k$ for z on $\overline{C}$. Conversely, under a mild hypothesis on C and $\Omega(x)$, it is proved that if rational functions $r_n(z)$ all of whose poles lie on or exterior to C_A, $A>1$, exist such that $|f(z)-r_n(z)|\leqq\Omega(n)/n^k$ for z in $\overline{C}$, then $f(z)$ is analytic interior to C and continuous on $\overline{C}$; $f^{(k)}(z)$ exists and is continuous on $\overline{C}$ with modulus of continuity $\omega(\delta)\leqq L[\delta\int_a^{a/\delta}\Omega(x)dx+\int_{1/\delta}^\infty[\Omega(x)/x]dx]$, where $a>1$ is a constant independent of δ. (Received August 21, 1951.)

16*t*. Abolghassem Ghaffari: *Laguerre polynomials satisfying a certain functional equation.*

This paper endeavors to find the most general solution of the functional equation (1) $f(x, s; y, t)=\int_V f(x, s; z, u)f(z, u; y, t)dz$, $s<u<t$, where f is Lebesgue-measurable in V. By applying Fréchet's method (Proc. London Math. Soc. vol. 39 (1935) pp. 515–540) one is led to take for the solution of (1) the series (2) $f(x, s; y, t)=\sum_{n=0}^\infty A_n(x, s)B_n(y, t)$, where A_n, B_n form a complete biorthonormal set of functions over V. Using this method and taking for the region V the interval $(0, +\infty)$, one finds that (3) $f(x, s; y, t)=\sum_{n=0}^\infty \psi_n(x)\psi_n(y)\theta^n(s, t)$, where the infinite sequence of functions $\psi_n(x)=\Gamma^{1/2}(n+1)\Gamma^{-1/2}(n+\alpha+1)x^{\alpha/2}\exp(-x/2)L_n^{(\alpha)}(x)$ form a complete orthonormal set over $(0, +\infty)$, $\theta(s, t)=a(s)/a(t)$ where $a(s)$ is a positive increasing function $\neq 0$, and $L_n^{(\alpha)}(x)$ is the nth Laguerre polynomial. It is shown that for s, t fixed such that $s<t$, and x, y varying arbitrarily, the series (3) is absolutely convergent in $(0, +\infty)$ and uniformly convergent over $(\epsilon, +\infty)$, ϵ being an arbitrary small but fixed positive number. It is proved that for $s=t$ and $x\neq y$ the series (3), being divergent, is summable uniformly in $(0, +\infty)$ by the method of arithmetic means with *zero sum* everywhere; and that for $s=t$, $x=y$, the series (3) diverges essentially and becomes *infinite*. The author has shown that the solution (3) is different from those obtained by A. Kolmogoroff (Math. Ann. vol. 104 (1931) pp. 415–458). (Received September 4, 1951.)

17*t*. R. V. Kadison: *Infinite unitary groups.*

The uniformly closed invariant subgroups of the unitary groups of the various factors are determined. It is shown that the unitary groups of factors of type II_1

and III are topologically simple. The uniform closure $G_f^{(1)}$ of the set of unitary operators which act as the identity on the complement of some subspace of finite relative dimension is a closed, proper, invariant subgroup of the unitary groups of factors of type I_∞ (all bounded operators) and factors of type II_∞. All such subgroups are obtained as direct products of $G_f^{(1)}$ with closed subgroups of the scalars $\{\lambda I; |\lambda| = 1\}$. (The group $G_f^{(1)}$ is itself topologically simple but is not algebraically simple.) It follows, for example, that finite products of self-adjoint unitary operators, or unitary operators with three points in their spectrum, etc., contained in a factor, lie uniformly dense in the group of all unitary operators in the factor. (Received September 4, 1951.)

18. M. S. Klamkin: *Asymptotic sums of series of the form* $\sum_{(b_r \neq b_s)=1}^{N} (\prod_{r=1}^{p} a_{b_r})^{-1}$.

Asymptotic sums of the series $\sum_{(a_r \neq a_s)=1}^{N} (a_r a_s)^{-1}$, $\sum_{(a_r \neq a_s)=1}^{N} (a_r a_s a_t)^{-1}$, and $\sum_{(a_r \neq a_s)=1}^{N} (a_r a_s a_t a_u)^{-1}$ are derived, assuming that $\sum_1^\infty (a_r)^{-1}$ diverges, by elementary means. Each of the given series is rearranged and expressed in terms of various sums which have already been derived in the author's previous paper, *Sums and asymptotic sums of series related to the harmonic series*. The results are then specialized for the case of $a_r = 1/r$. (Received September 13, 1951.)

19*t*. R. T. Leslie and E. R. Love: *An extension of Mercer's theorem.*

Mercer's summability theorm is generalised to the extent of replacing the arithmetic means by the means of any regular, or merely convergence-preserving, method of summation. The relations between this extension and some known converses of Toeplitz's theorem, due to R. P. Agnew, are discussed in detail. The proof of the theorem rests on an extension of an inequality of W. A. Hurwitz for the oscillation of Toeplitz means of a bounded sequence. (Received September 24, 1951.)

20*t*. J. S. MacNearney: *Continued fractions in which the elements are operators.* I. *The J-sequence, associated linear equations, and the general continued fraction.*

This paper is designed to lay the foundations for a generalization of some of the classical studies of continued fractions. Continued fractions over an additive Abelian group A are defined so that the elements and the approximants are operators, here taken to be homomorphisms or isomorphisms among the subgroups of A. Analogues of numerators and denominators, even and odd parts, and equivalence transformations of continued fractions are discussed. Linear fractional transformations are defined among sets of operators in such a way that continued fractions over A are generated by sequences of these transformations. Since this paper is concerned primarily with algebraic relationships which are consequences of the definitions and no questions of convergence are considered, no topology is assumed in the group A. It is anticipated that subsequent reports will require further that A be, for example, a metric linear space or a normed vector space over the complex numbers. An appendix indicates how this theory includes the ordinary notion of a continued fraction and, also, how it includes a certain generalization of Jacobi-matrices (see, for examples, H. S. Wall, *J-matrices of interior order k*. Preliminary report, Bull. Amer. Math. Soc. Abstract 57-2-137). (Received August 29, 1951).

21*t*. J. L. Massera: *Conditional stability of homeomorphisms.*

Let T be a homeomorphism of the neighborhood $\mathfrak{W}$ of the origin o of a Banach space $\mathfrak{Z}$ onto a neighborhood of o, $To=o$; $\mathfrak{Z}$ is the product of two Banach spaces $\mathfrak{X}$, $\mathfrak{Y}$ and any point of $\mathfrak{Z}$ will be denoted $z=(x, y)$, $x\in\mathfrak{X}$, $y\in\mathfrak{Y}$. Write $z_1=Tz_0$ and assume that T is given by the equations $x_1=Lx_0+\phi_1 z_0$, $y_1=My_0+\phi_2 z_0$, L, M linear, $\|\phi_i z\|/\|z\|\to 0$ as $z\to o$, $\|Lx\|\leqq\lambda\|x\|$, $\|My\|\geqq\mu\|y\|$, $0\leqq\lambda<1$, $\lambda<\mu$. Then, if dim $\mathfrak{Y}<\infty$, a set of stability $\mathfrak{E}\subseteq\mathfrak{W}$ exists with the properties: (i) $o\in\mathfrak{E}$; (ii) $T\mathfrak{E}\subseteq\mathfrak{E}$; (iii) if $z\in\mathfrak{E}$, $T^n z\to o$ as $n\to+\infty$; (iv) $\mathfrak{E}$ is tangent at o to the space $\mathfrak{X}$; (v) the projection of $\mathfrak{E}$ in $\mathfrak{X}$ coincides with the projection of $\mathfrak{W}$ in $\mathfrak{X}$; (vi) any set of stability having properties (i)-(iv) is a part of $\mathfrak{E}$. If moreover T is differentiable at any point of $\mathfrak{W}$, the differential being continuous at o (if the differential is continuous at any point of $\mathfrak{W}$; if T is analytic), then $\mathfrak{E}$ is the graph of a transformation $y=Ex$ having a differential continuous at o (continuous in a neighborhood of o; E is analytic). In the latter cases the assumption dim $\mathfrak{Y}<\infty$ may be replaced either by dim $\mathfrak{X}<\infty$ or by the assumption that T^{-1} is differentiable at o. (Received September 10, 1951.)

22. C. N. Moore: *On the theory of patterns and its application to the prime pair problem.*

If we stop the procedure known as the sieve of Eratosthenes at the end of a finite number of steps, the successive differences of the integers remaining constitute a set of even integers in which a certain initial sequence is indefinitely repeated. Such a sequence may be designated as a pattern. By making use of the pattern 6, 4, 2, 4, 2, 4, 6, 2 resulting from stopping the sieve after the removal of multiples of five, an estimate may be obtained of the extent to which a further continuation of the sieve process eliminates the differences two in the earlier portions of later patterns. This enables one to infer the infinitude of prime pairs. (Received September 12, 1951.)

23. Leo Sario: *Existence of functions as a boundary property.*

Denote by α, β, γ, δ the classes of nonconstant harmonic functions (integrals) with following additional properties: α all, β bounded, γ half-bounded, δ Dirichlet bounded functions. Let α_0, β_0, δ_0, γ_0 be the corresponding subclasses of functions with vanishing periods along dividing cycles and a, b, g, d the subclasses of single-valued harmonic functions. Classes of analytic functions are denoted by couples of letters, corresponding to the real and imaginary part respectively. Consider an arbitrary Riemann surface R and a compact subregion D, bounded by a finite set l of simple analytic curves. In the boundary neighborhood $N=R-D$ let L_0 be the normal linear operator which furnishes the harmonic function with the minimal Dirichlet integral among functions with given harmonic boundary values on l. *There are functions ff' (1) for $f=\beta_0$, γ_0, δ_0 and $f'=\alpha$, α_0 on every R of positive (finite or infinite) genus; on a planar R there are these functions only if R is hyperbolic; (2) for $f=b$, g, d and $f'=\alpha$, α_0 on any R if and only if some $\phi+i\bar{\phi}\in ff'$ in N, subject to $\int_l d\bar{\phi}=0$, satisfies $\phi\not\equiv L_0\phi$; (3) for $f=b$, d and $f'=a$, b, d on any R of finite genus if and only if some $\phi+i\bar{\phi}\in ff'$ in N fulfils $\phi\not\equiv L_0\phi$.* (Received August 27, 1951.)

24. L. V. Toralballa: *Some theorems about functions in a general field with valuation.*

Some classical theorems in the theory of functions of a complex variable are proved to hold in a general field with valuation. Two of these theorems are: 1. Let G be a field with valuation which is complete in the topology induced by this valuation. Let f be a function on G to G such that its derivative is defined and is equal to zero at each point of E where E is a rectifiable arc-wise connected open sub-set of G. Then f is a constant over E. Let G be as in "1," and let C be a rectifiable curve in G. C is said to be locally arc-wise regular at P when there exists a sphere, $S(P, \epsilon)$, such that the set of all ratios arc length ab/distance ab (where a is not equal to b; a, b are elements of C as well as of the sphere, $S(P, \epsilon)$) is bounded. We then have: 2. Let the curve C be arc-wise regular at z_0. Let f be integrable over C, and let $F(z)$ be $\int_a^z f(z)\,dz$ where a and z are points of C. Then $F(z)$ is continuous at z_0. If, in addition, $f(z)$ is continuous at z_0, then $F(z)$ is differentiable at z_0 and $F'(z_0) = f(z_0)$. (Received September 13, 1951.)

25. E. H. Zarantonello: *On trigonometric interpolation.*

Let $P(t)$ be a trigonometric polynomial of degree n, $P^{(r)}(t)$ its rth derivative, and $P^{[r]}(t)$ its rth divided difference $\Delta^r P(t)/\Delta t^r$ with regard to a constant increment $\Delta t = 2\pi/m$. If $m > 2n$ and $1 < p < \infty$, then $A \leqq \{\int_0^{2\pi} |P^{(r)}(t)|^p dt\}^{1/p} / \{\sum_1^m |P^{[r]}(t_k)|^p \cdot \Delta t_k\}^{1/p} \leqq A_{p,r}$, where the t_k are m equidistant points modulo 2π, $A_{p,r}$ a constant depending on p and r only, and A an absolute constant. For $r = 0$ ($P^{(r)} = P^{[r]} \equiv P$) this reduces to a previous result due to J. Marcinkiewicz (*Sur l'interpolation*, Studia Mathematica vol. 6 (1936)). As a consequence of the above inequalities, it follows that if $f(t)$ is a periodic function with a p-integrable rth derivative and $P(t)$ a trigonometric polynomial interpolating f at equidistant points, then $\{\int_0^{2\pi} |P^{(r)}|^p p dt\}^{1/p} \leqq A_{p,r} \{\int_0^{2\pi} |f^{(r)}|^p dt\}^{1/p}$ for $1 < p < \infty$. Furthermore, the approximation of f by P is $O((\Delta t)^{r-1/p})$ for the uniform distance and $O((\Delta t)^r)$ for the p-norm distance. Applications are made to the discretization of some totally continuous functional equations appearing in conformal mapping and in the theory of free boundaries in hydrodynamics. (Received September 13, 1951.)

Applied Mathematics

26. Gertrude Blanch: *On the numerical solution of parabolic partial differential equations.*

The equation treated has the form $(\partial u/\partial t) = (\partial^2 u/\partial x^2) + f(x, t, u)$, with continuous initial and boundary conditions over a closed region of the $x-t$ plane. To solve the equation by finite difference approximations, a lattice is introduced with intervals h and k in the x and t-directions, respectively. It is known that if the finite difference approximation is of order two, then the latter converges to $u(x, t)$ as h approaches zero, provided the mesh ratio, k/h^2, is no greater than 1/2. For at least some difference approximations of order four, the mesh ratio can be as large as 2/3. The primary objective of the present study is to seek that mesh ratio, for a given difference approximation, which will lead to results that are within a given upper bound of error in the solution with the least amount of work. It is shown that the largest admissible mesh ratio is not always the most economical one, and that much depends on the function $f(x, t, u)$, the initial, and the boundary conditions. Five numerical examples are given to illustrate the analysis. In the process of computing, the technique recommended by Hartree and Womersley of extrapolating an improved solution from two difference approximations was tried with considerable success. (Received September 13, 1951.)

27. W. C. Carter and G. L. Spencer: *On the numerical solution of hyperbolic systems of partial differential equations with two characteristic directions.*

To find bounds for the truncation error in the numerical solution by difference methods of the Cauchy problem for hyperbolic systems of partial differential equations, a careful examination is made of the details of H. Lewy's proof of convergence of the difference process to the actual solution (Math. Ann. vol. 98 (1927) pp. 179–191). The conditions for the existence of a solution are relaxed, and it is shown that the sequence of numerical solutions corresponding to decreasing mesh size is uniformly convergent. By considering the properties of a subsequence of difference solutions corresponding to mesh sizes $h/2^n$, h fixed, extrapolation formulae giving improved numerical results are developed. Various approximation methods are considered, and examples are given in the case of the supersonic axisymmetric irrotational flow of a compressible perfect fluid. (Received September 10, 1951.)

28t. E. A. Coddington: *The stability of infinite differential systems associated with vortex streets.*

Consider the k-fold infinite linear system of differential equations (L): $d\rho/dt = A\rho$, where A is a matrix whose elements $A_{n,m}$, $n, m=0, \pm 1, \pm 2, \cdots$, are $k\times k$ matrices of complex constants and $A_{n,m}=A_{n-m}$. If the elements of the matrix A_p are the Fourier constants of continuous functions on $[0, 1]$, one gives an integral representation of the unique analytic solution of (L) for any initial $\rho^0=(\rho_n^{0i})\in H_2$, i.e., $\|\rho^0\|=(\sum_{n=-\infty}^{\infty} \sum_{i=1}^{k} |\rho_n^{0i}|^2)^{1/2}<\infty$. For $k=1$ this was treated by Wintner. The system (L) is stable relative to $\rho^0\in H_2$ if for every solution ρ with this initial ρ^0, $\limsup_{t\to\infty} \|\rho\|<\infty$. Necessary and sufficient conditions for stability are given. An application is made to a problem of von Kármán (Nach. Ges. Wiss. Göttingen (1912) pp. 547–556) concerning parallel rows of vortices. The linearized problem resulting from perturbing two rows of vortices at $(2nb+c, a)$, $(2nb-c, -a)$ is of type (L). If $r=a/b$, $q=c/b$, this system is shown to be unstable for $q\neq 1/2$. Contrary to previous beliefs, *instability* also occurs when $q=1/2$ even if r satisfies the necessary condition $\cosh^2 \pi r=2$. Further applications are considered (Received August 15, 1951.)

29t. Abolghassem Ghaffari: *Supersonic flow and simple waves.*

This paper integrates, without using the theory of characteristics, the equations of plane irrotational flow of a perfect gas in the case when the hodograph is degenerated. Taking into account the conditions of continuity, zero rotation, and Bernoulli's equation, together with the fundamental assumption, which implies that the components of fluid velocity, u and v, are each a function of fluid speed w only, it is found that $(du/dw)^2+(dv/dw)^2=w^2/a^2$, where a is the local speed of sound. Moreover if $u=-w \sin \phi$ and $v=w \cos \phi$, where ϕ is the angle between w and y-axis as in Meyer's notation (G. Taylor and W. Maccoll, *Aerodynamics theory*, ed. by Durand, vol. 3, div. H, chap. IV, 1935), one finds that $(d\phi/dw)^2=(w^2-a^2)/w^2a^2$, so that the motion is necessarily *supersonic*. It is shown that the lines of equal speed, pressure (isobar) and density are *all straight lines*, although they are not necessarily concurrent as in the original Prandtl-Meyer theory. Furthermore if θ is the angle of slope of isobar and $\psi=\theta-\phi$, it is found that $\phi=\lambda^{-1} \tan^{-1} (\lambda \tan \psi)-\psi$, where $\lambda=(\gamma-1)^{1/2}(\gamma+1)^{-1/2}$ and γ is the adiabatic index. The generalization of the Prandtl-Meyer *wedge-solution* for the flow past an arbitrary fixed boundary is obtained analytically. (Received September 4, 1951.)

30. L. E. Payne: *The flow about bodies of revolution and generalized electrostatics.*

The irrotational symmetric flow of an ideal incompressible fluid about certain bodies of revolution is obtained by the method of generalized electrostatics developed by Weinstein (Proceedings of the Fourth Symposium in Applied Mathematics, American Mathematical Society, *Fluid dynamics*, 1951, in print). This method makes possible the determination of the flow about a lens without requiring the multi-sheeted Riemann space employed by Shiffman and Spencer (Quarterly of Applied Mathematics vol. 5 (1947) pp. 270–288). The stream function for the lens is found to be expressible in terms of Legendre functions of the Mehler type. The method of generalized electrostatics is also used in the determination of the flow about various profiles in plane hydrodynamics. (Received September 12, 1951.)

31. H. F. Weinberger: *The intermediate problem and error estimation in the method of Weinstein.*

In a paper previously presented to the Society (Bull. Amer. Math. Soc. Abstract 57-6-514), the author gave an estimate of the error in the upper bounds for eigenvalues of an operator L in a space $\mathfrak{L}\subset\mathcal{H}$ obtained by the method of Weinstein. For this estimate, it was necessary to use the particular nth intermediate problem in the space $\mathcal{H}\ominus(p_1, \cdots, p_n)$ where p_i is the projection into the subspace $\mathcal{P}=\mathcal{H}\ominus\mathfrak{L}$ of the ith eigenvector of L in $\mathcal{H}$. It is found in applications, however, that it may be possible only to approximate p_i by another vector f_i, rather than to evaluate it exactly. For such cases, an estimate of the difference between the eigenvalues of L in $\mathcal{H}\ominus(p_1, \cdots, p_n)$ and in $\mathcal{H}\ominus(f_1, \cdots, f_n)$ with $f_i \cong p_i$ is obtained. This, then, gives upper and lower bounds of the eigenvalues of L in $\mathfrak{L}$ in terms of the eigenvalues of L in $\mathcal{H}\ominus(f_1, \cdots, f_n)$. The latter can be evaluated by means of the Weinstein determinant. (Received September 12, 1951.)

32t. L. A. Zadeh and K. S. Miller: *Generalized ideal filters.*

The notation $v=Nu$ signifies that $v(t)$ is the response of a network N (linear or nonlinear) to a signal $u(t)$. N_2N_1 represents a tandem combination of N_1 and N_2, with N_1 preceding N_2. Consider two classes of signals $\mathfrak{M}$ and $\mathfrak{N}$ both of which contain the null element. A network N is called an ideal filter if $N(u+v)=u$ for all $u\in\mathfrak{M}$ and $v\in\mathfrak{N}$. ($\mathfrak{M}$ and $\mathfrak{N}$ are called, respectively, the acceptance and rejection manifolds of N.) An immediate consequence of this definition is that any ideal filter is idempotent. The converse, however, is true only for linear filters. If N is an ideal filter (linear or nonlinear) and L is a nonsingular linear network, then LNL^{-1} is an ideal filter whose acceptance and rejection manifolds are $L(\mathfrak{M})$ and $L(\mathfrak{N})$ respectively. Physical applications of ideal filters to the filtration and simultaneous transmission of signals are considered. (Received September 5, 1951.)

Topology

33. W. H. Marlow: *Limits of functions on directed sets.* I. Preliminary report.

If α is on the set of cofinal subsets of a directed set X to 2^A, A a given set, then $L_*\alpha=\bigcap\alpha(X_c)$, intersection for all cofinal subsets X_c of X, and $L^*\alpha=\bigcap\alpha(X_r)$, intersection for all residual subsets X_r, are defined to be the *lower* and *upper limits* of α, respectively. (See J. Tukey, *Convergence and uniformity in topology*, Princeton, 1940, for meaning of 2^A, cofinal and residual.) If α and β are *monotone nondecreasing* ($X_{c'}\subseteq X_c$ implies $\alpha(X_{c'})\subseteq\alpha(X_c)$), a typical result involving the Boolean function $\alpha\cup\beta$ is: (1) $L_*(\alpha\cup\beta)\subseteq(L_*\alpha)\cup(L^*\beta)$. A partition of the points in $L^*(\alpha\cup\beta)$ according to membership in the limits of α and β and $L_*(\alpha \cup \beta)$ is given and some properties of *convergence* (when $L_*\alpha = L^*\alpha$) are developed. For f on X to 2^Z, Z a space with *topological function* "t" (merely a function on 2^Z to 2^Z), the *lower* and *upper limits of f relative to t* are defined by replacing α by the function whose value at the place X_c is $t(\{f(x_c) \mid x_c \in X_c\})$ in the definitions of $L_*\alpha$ and $L^*\alpha$, respectively. The previous results apply if $t(A \cup B) = tA \cup tB$ or, in some cases, if t is only monotone nondecreasing. Then (1) specializes to limits of different functions on X or to limits of a single function relative to different topological functions. (Received September 13, 1951.)

34*t*. W. H. Marlow: *Limits of functions on directed sets.* II. Preliminary report.

This is a continuation of Part I. After E. W. Chittenden, the upper limit of f on X to 2^Z relative to a Kuratowski closure function is called the *Peano limit* of f, Lf. Since f and the function whose value at the place $x \in X$ is $\{f(x') \mid x' > x\}$ have the same Peano limits, properties of Lf, f arbitrary, follow from results for the case: $x' > x$ implies $f(x') \subseteq f(x)$. Conditions are obtained which are both necessary and sufficient for $(Lf) \cap A \neq 0$, $A \subseteq Lf$, $Lf \subseteq A$, and $Lf = A$ when A is closed in a compact Hausdorff space. A dominance relation for functions on directed sets is introduced and properties of directed families of open sets are exploited. Of central importance are functions whose domains contain residual subsets upon which the null set is not a value. Functions on directed sets to compacta are treated and an extension of the "metric convergence" of Hausdorff (*Mengenlehre*, Berlin, 1935) is made. Results of this paper are shown to be related to those of E. H. Moore and H. L. Smith, G. Birkhoff, and others. (Received September 13, 1951.)

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