

$bx - ay = 1$  for all pairs of integers  $a, b$  such that  $1 \leq a \leq b \leq 1025$  and  $(a, b) = 1$ , where the entries are arranged in order of magnitude of the ratio  $a/b$ . Of course it is possible to extend the usefulness of the table well beyond its apparent range.

Since the main table of the work under review does not give the decimal equivalents of the fractions listed, there is considerable difficulty in locating a given fraction in the series or in fixing a given irrational number (or rational number with denominator greater than 1025) between two fractions of the series. While it would clearly be out of the question to give the decimal equivalent of every fraction listed, it seems to the reviewer that it would have been quite feasible to give at the end of each line of the table the decimal equivalents of the last pair of fractions in the line. This would have enhanced the value of the table considerably, for it would have made the location problem relatively easy. As it is, the user of the table is expected to locate a given fraction in the series (or to determine in what interval a number not in the series would fall) by appealing to the uniform distribution of the Farey fractions. (Cf. problem 189 of section II of Pólya and Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Berlin, Springer, 1925.) To be sure, the author gives a table of the locations of the fractions equivalent to  $k/1000$  ( $k=0, 1, 2, \dots, 500$ ), but between two consecutive such fractions there are on the average about 320 other fractions.

This work is the first volume in a series of mathematical tables which is to be published by the Royal Society and which is intended as a continuation of the well known series of tables published by the British Association. The format and printing of this first volume are very satisfactory. Although this work will be a mere curiosity to most mathematicians and will certainly not have widespread use, number-theoretic experimenters will find it of considerable interest.

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*A theory of cross-spaces.* By Robert Schatten. (Annals of Mathematics Studies, no. 26.) Princeton University Press, 1950. 7+153 pp. \$2.50.

The subject of this book is the generalization to Banach spaces of the construction of the algebraic direct product of two finite-dimensional vector spaces. The main problems arise from the variety of possible norms which can be used in the algebraic product space and the resulting profusion of Banach spaces which have honest claim to the title of direct product space. The results, mostly due to the author and to J. von Neumann, are outlined in a long introductory chapter; in the next paragraphs we state briefly the argument of

the text. The author has provided a clear exposition of the carefully limited topic he set himself.

Given two Banach spaces,  $B_1$  and  $B_2$ , with conjugate spaces  $B_1^*$  and  $B_2^*$ , the algebraic direct product  $B_1 \circ B_2$  consists of suitable equivalence classes of the set of all formal sums of formal products,  $\sum_{i \leq n} f_i \otimes g_i$ ,  $f_i \in B_1$ ,  $g_i \in B_2$ ; the definition of equivalence is so chosen that the correspondence of  $\sum_{i \leq n} f_i \otimes g_i$  with the operator  $A$  from  $B_1^*$  to  $B_2$  defined by  $A(F) = \sum_{i \leq n} F(f_i)g_i$  is an isomorphism between  $B_1 \circ B_2$  and a linear subspace of the space  $L(B_1^*, B_2)$  of linear bounded operators from  $B_1^*$  to  $B_2$ . Let  $\lambda(\sum_{i \leq n} f_i \otimes g_i)$  equal the norm of the corresponding  $A$ .

A norm  $\alpha$  in the linear space  $B_1 \circ B_2$  is called a crossnorm if  $\alpha(f \otimes g) = |f| |g|$ . It is shown that  $\lambda$  is a crossnorm and that there is in  $B_1 \circ B_2$  a greatest crossnorm  $\gamma$ .

Each element of  $B_1^* \circ B_2^*$  determines a linear functional on  $B_1 \circ B_2$  by  $\phi \mapsto \sum_{j \leq m} F_j \otimes G_j$  if  $\phi(\sum_{i \leq n} f_i \otimes g_i) = \sum_{j \leq m} \sum_{i \leq n} F_j(f_i)G_j(g_i)$ . Hence every crossnorm  $\alpha$  on  $B_1 \circ B_2$  determines an associate norm  $\alpha'$  on  $B_1^* \circ B_2^*$ .  $\lambda$  is the least crossnorm whose associate is also a crossnorm. On  $B_1^* \circ B_2^*$ ,  $\gamma'$  and  $\lambda$  are equal.

Completion of  $B_1 \circ B_2$  under crossnorm  $\alpha$  gives the cross-space  $B_1 \otimes_\alpha B_2$ . If  $\alpha \geq \lambda$ , completion of  $B_1^* \circ B_2^*$  under the associate crossnorm  $\alpha'$  yields the associate space  $B_1^* \otimes_{\alpha'} B_2^*$ ; the natural embedding of this associate space into the conjugate space  $(B_1 \otimes_\alpha B_2)^*$  is an isometry but need not fill up all of  $(B_1 \otimes_\alpha B_2)^*$ . A representation of  $(B_1 \otimes_\alpha B_2)^*$  as a renormed linear subspace of  $L(B_1, B_2^*)$  is obtained; in particular,  $(B_1 \otimes_\gamma B_2)^*$  is all of  $L(B_1, B_2^*)$  with the usual norm.

Specialization of  $B_1$  and  $B_2$  to two copies of Hilbert space,  $H$  and  $H^*$ , gives results on the ideal of completely continuous operators on  $H$ .  $H \circ H^*$  can be interpreted as the set of operators from  $H$  to  $H$  of finite rank. If  $E$  is the set of all operators  $X \in L(H, H)$  such that  $\sigma(X) = (\sum_i |X\phi_i|^2)^{1/2} < \infty$  for a fixed complete orthonormal set  $\{\phi_i\} \in H$ , then  $\sigma$  is a crossnorm on  $H \circ H^*$  and  $E$  with the norm  $\sigma$  is equivalent to  $H \otimes_\sigma H^*$ . Similarly, the space  $H \otimes_\lambda H^*$  can be interpreted as  $I$ , the ideal of all completely continuous operators belonging to  $L(H, H)$ . The conjugate and associate spaces of  $H \otimes_\lambda H^*$  coincide with  $H \otimes_\gamma H^*$ , which, in turn, can be represented as the closure of  $H \circ H^*$  under the norm  $m(A) = t(|A|)$ , where  $|A| = (A^*A)^{1/2}$  and  $t(X)$ , the trace of  $X$ ,  $= \sum_i (X\phi_i, \phi_i)$ ,  $\{\phi_i\}$  a complete orthonormal system. Finally,  $I^{**}$  is equivalent to  $(H \otimes_\gamma H^*)^*$  which is equivalent to  $L(H, H)$ .

Appendices discuss properties of a crossnorm  $\alpha$  which are connected with equality of conjugate and associate spaces.

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