SOME REMARKS ABOUT LIE GROUPS TRANSITIVE ON SPHERES AND TORI

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The present note pertains principally to two papers of D. Montgomery and H. Samelson [**1**, **2**],[1] in which the authors study compact Lie groups transitive on tori [**1**] and spheres [**2**]. I will here prove in another way, generalize, and sharpen a part of their results. §1 contains the remarks to [**1**], §2 to [**2**]; they are independent of one another and the methods used in both are quite different.

I recall first the definition and some simple properties of homogeneous spaces. A manifold W is a homogeneous space under the Lie group[2] G if to each element a of G there corresponds a differentiable transformation $T_a:x\to T_a(x)$ of W into itself such that:

(1) $T_a(x)$ depends continuously on the pair $a\in G$, $x\in W$.

(2) To the product (ab) corresponds the mapping $x\to T_{(ab)}(x) = T_a[T_b(x)]$.

(3) Given any two points x, y in W, there exists $a\in G$ such that $T_a(x)=y$ (that is, G is *transitive* on W).

G is said to be *effective* on W if only the identity element e of G induces the identity transformation of W.

Let us choose an arbitrary point x of W. The set of elements h in G for which $T_h(x)=x$ is a closed subgroup H of G, called the *associated group*. As is well known [**3**, no. 29], W may be identified with the space of left cosets G/H, the mappings T_a being then: $xH\to(ax)H$. Actually, H depends on the choice of $x\in W$ and should be denoted H_x, but I shall in general drop the index x as there will be no danger of confusion and also because all the groups H_x $(x\in W)$ are conjugate to each other in G.

When considering a homogeneous space as the space of left cosets, it is quite easy to prove that *every subgroup of H which is invariant in G induces the identity mapping of W*, and, conversely, *a subgroup of H, each element of which induces the identity of W, is invariant in G.*

1. The n-dimensional torus as a homogeneous space.

In [**1**], D. Montgomery and H. Samelson proved that a Lie group which acts transitively and effectively on the n-dimensional torus is itself the n-dimensional toral group T^n. Actually, as they remark at the end of

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The manifolds and Lie groups considered here are always *compact*.

their note, their proof gives at the same time the stronger theorem:

Let W be an n-dimensional homogeneous space under a compact connected Lie group G, the first Betti number of W being n.

Then W is homeomorphic to the n-dimensional torus, and if G is effective on W, it is isomorphic to T^n.[3]

I shall prove here the more general theorem:

THEOREM I. *Let W be an n-dimensional homogeneous space under the compact connected Lie group G. Let us suppose that for one index j $(1 \leqq j \leqq n-1)$ the jth Betti number of W equals the binomial coefficient $C_{n,j}$. Then:*

(a) *W is homeomorphic to the n-dimensional torus;*

(b) *if G is effective on W, G is isomorphic to the n-dimensional toral group T^n.*[3]

The demonstration is quite different from that given in [**1**] in the case $j=1$, and employs the theory of integral invariants on a homogeneous space [**4**], the main theorems of which I review now.

Let us denote by p_j the jth Betti number of W and by n_j the number of linearly independent differential exterior forms of degree j on W which are invariant under all transformations of G. Then we always have:

$$p_j \leqq n_j \leqq C_{n,j}.$$

The first inequality follows from the theorems of G. de Rham [**5**] and from the fact that every closed form is equivalent to an invariant one [**4**, Theorem I]. To obtain the second inequality one needs only to remark that an invariant form is completely determined by its value at one point of W.

Let now x_0 be a definitely chosen point of W, $H=H_{x_0}$ the associated group; we can take in a neighborhood $U(e)$ of e in G canonical coordinates $x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_{n+s}$ such that $H \cap U(e)$ is the s-plane of the last s coordinates; $x_1, \cdots, x_{n+s}$ may also be taken as coordinates in the tangent space to G at e. The transformations: $x \rightarrow (a^{-1}xa)$, where $a \in G$, $x \in U$, are linear, and form the adjoint linear group of G, which I shall denote by Ad G. G being compact, Ad G may be assumed to be orthogonal. The representation of H contained in Ad G splits then into two parts, one of which is a linear group γ leaving invariant the set of variables $x_1, \cdots, x_n$. But $x_1, \cdots, x_n$ can be taken as coordinates in a neighborhood $V(x_0)$ of x_0 in W (or

[3] G being then abelian, H reduces to the identity element if G is effective; W may be identified with the manifold of G.

as coordinates in the vector space $L(x_0)$ tangent to W at x_0), so that γ indicates how H acts on $V(x_0)$ (or on $L(x_0)$).

To γ there corresponds a linear group γ_j of degree $C_{n,j}$: the group of transformations of j-dimensional elements of $L(x_0)$ induced by the operations of γ. The following theorem allows us to compute, at least theoretically, n_j with the help of γ_j (see [**4**, nos. 25, 28]).

The number of linearly independent invariant differential forms of degree j equals the number of times the trivial representation[4] *of H occurs in* γ_j.

If W is the manifold of a group G', one takes as transformation group of W the left and the right translations of G'; then $G=G'\times G'$, and the differential forms invariant under G are the doubly (left *and* right) invariant forms. The associated group H_e is isomorphic to Ad G' and the number n_j of doubly invariant independent forms is also given by the previous theorem, where γ is replaced by Ad G' and γ_j by the corresponding group (Ad $G')_j$ of transformations of j-dimensional elements (see [**4**, no. 53]).

Theorem I will be an immediate consequence of the results mentioned above and of the following rather trivial lemma:

LEMMA. *Let A be a regular $n\times n$ matrix, A_j the matrix of degree $C_{n,j}$ giving the transformation of j-dimensional planes induced by A.*

If for one index j $(1\leqq j\leqq n-1)$ $A_j=E$ (identity matrix), then $A=\pm E$.

PROOF. The coefficients of A_j are the determinants of degree j of A, and especially the diagonal terms of A_j are the principal j-minors of A.

If $A\neq cE$, then there is at least one vector $\vec{x}$ which is not eigenvector of the linear transformation: $\vec{x}\rightarrow A\vec{x}$ given by A, that is, $\vec{x}$ and $A\vec{x}$ are linearly independent. Let π_j be a j-dimensional plane containing $\vec{x}$ but not $A\vec{x}$ (such a plane exists, since $j\leqq n-1$). π_j is certainly not invariant under A_j and $A_j\neq E$, which contradicts the assumption. Therefore we must have $A=cE$; but then each diagonal term of A_j equals c^j; if $A_j=E$, one has $c=\pm 1$ and $A=\pm E$.

PROOF OF THEOREM I. Let W be a n-dimensional homogeneous space, one Betti number p_j of which equal $C_{n,j}$. Then we know that $n_j=C_{n,j}$ and that γ_j reduces to the identity matrix. The previous lemma shows that γ consists either of E or of $+E$ and $-E$. In the former case, every element of the associated group H induces the identity mapping of a neighborhood $V(x)$ of x in W, and therefore

[4] That is, the representation of degree 1 which assigns the number one to each element of H.

on the whole of W.[5] H is then invariant in G and W is homeomorphic to the manifold of a group $G' = G/H$. We also see that, if G is effective on W, $H = \{e\}$ and $G' = G$.

In the second case $(-E \in \gamma)$, H possesses a subgroup H_1 of index two represented by $+E$ in γ. H_1 is invariant in G, $\overline{W} = G/H_1$ is the manifold of a group G' and $p_j(\overline{W}) \leqq C_{n,j}$. But on the other hand $\overline{W}$ is a two-fold covering space of W and therefore, as is known, $p_j(\overline{W}) \geqq p_j(W)$. Thus $p_j(\overline{W}) = C_{n,j}$.

We know now that, if $p_j(W) = C_{n,j}$, then W is either homeomorphic to or twice covered by the manifold of a group G', and that $G = G'$ if G is effective on W. The latter case cannot occur when G' is abelian (see footnote 3).

We have seen that $p_j(G') = C_{n,j}$. Theorem I will therefore be completely proved if we establish the proposition:

Let W be the manifold of a compact connected n-parameter Lie group G. For one index j $(1 \leqq j \leqq n-1)$ let $p_j(W) = C_{n,j}$.

Then G is isomorphic to the n-dimensional toral group T^n.

PROOF. This could be deduced from theorems of E. Cartan and H. Hopf on the Poincaré polynomials of compact Lie groups, but we can also follow the same method as above: if $p_j = C_{n,j}$ then $n_j = C_{n,j}$, $(\mathrm{Ad}\, G)_j = E$ and $\mathrm{Ad}\, G = E$ ($\mathrm{Ad}\, G$ is connected and contains only one element if it is discrete). That means that $(a^{-1}xa) = x$ for $a \in G$, $x \in U(e)$, and therefore also for every $x \in G$, since an element of a connected topological group may be written as the product of a finite number of elements taken in an arbitrary neighborhood $U(e)$ of the identity. G is then abelian; being compact and connected, it is isomorphic to T^n according to a well known theorem [**3**, no. 43]).

2. **Even-dimensional spheres as homogeneous spaces.** In [**2**], Montgomery and Samelson study spheres of arbitrary dimensions; their results and demonstrations show that the cases of even and odd dimensionality have to be treated separately. Here I shall consider only the simpler one: even-dimensional spheres.

It is first shown in [**2**] that a compact connected Lie group acting transitively and effectively on an even-dimensional sphere S^n is simple. S^n being simply connected and having, for n even, an Euler-characteristic $\chi(S^n)$ equal to two, that theorem is contained in the following statement:

THEOREM II. *Let G be a compact connected Lie group acting transi-*

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[5] This last point may be for instance deduced from the fact that the transformations T_a are isometric mappings of W onto itself in a complete Riemannian metric [**3**, no. 36].

tively and effectively on a simply connected space W which has an Euler-characteristic equal to a prime number.

Then G is simple.

The proof is based on a theorem of H. Hopf and H. Samelson [6] which I shall formulate a little later, but first I must recall some points of the theory of compact Lie groups.

All maximal abelian subgroups of a compact connected Lie group are toral groups and conjugate to each other (see for example [6, no. 4]). Their common dimension r defines the *rank* $r(G)$ of G. Let T^r be a maximal toral group; the normalizer $N(T^r)$ of T^r (that is, the totality of elements $x \in G$ for which $x^{-1}T^r x \subset T^r$) has also the dimension r and consists of a finite number of cosets of T^r [6, Hilfssatz 2]; each coset defines one automorphism of T^r and the group $N(T^r)/T^r$ is isomorphic to the group of automorphisms of T^r obtained by means of the inner automorphisms of G leaving T^r invariant; this group plays a fundamental role in the theory of semi-simple Lie groups. I shall call it $\Phi(G)$; it is independent of the choice of T^r since all maximal toral groups of G are conjugate to each other. If H is a proper subgroup of G having the same rank as G, the group $\Phi(H)$ is of course a subgroup of $\Phi(G)$. *If H is a proper connected subgroup of same rank as G, then $\Phi(H)$ is a proper subgroup of $\Phi(G)$.* This is not explicitly stated, but follows easily from the theory of singular elements in a compact group (see, for example [7], especially §2, nos. 5, 7).

The theorem of Hopf and Samelson I need is:

Let W be a homogeneous space under a compact connected Lie group.

Then $\chi(W) \geqq 0$; it is positive if and only if the rank of the associated group H equals the rank of G; in that case, $\chi(W)$ is equal to the index of $\Phi(H)$ in $\Phi(G)$.[6]

Proof of Theorem II. Let W be a homogeneous space possessing the properties listed in Theorem II, and let H be the associated group; then $r(H) = r(G)$; moreover, W being simply connected, H is connected [3, no. 31],[7] and we see, by the way, that $\chi(W) > 1$. Let us call a connected subgroup of G *maximal* if it is not contained in another *connected* proper subgroup of G. Then, if $\chi(W)$ is a prime number, H is maximal, for if there were a connected group H' such that $H \subset H' \subset G$, $H \neq H' \neq G$, we should have $\Phi(H) \subset \Phi(H') \subset \Phi(G)$ with $\Phi(H) \neq \Phi(H')$

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[6] That is, the quotient of the order of $\Phi(G)$ by the order of $\Phi(H)$.

[7] In our special case, the converse is also true: *If H is connected and if $r(H) = r(G)$, then G/H is simply connected.* This follows from the fact that H contains a toral group T^r maximal in G and that every closed curve in G is homotopic to a closed curve in T^r.

$\neq \Phi(G)$ (see the previous paragraph), and the index of $\Phi(H)$ in $\Phi(G)$ could not be a prime number.

Let us suppose now that G is not simple. Then $G=\overline{G}/N$, where N is a finite group and $\overline{G}$ a direct product $G_1 \times G_2 \times \cdots \times G_k$ of compact simple groups [3, no. 52]; W may be considered in an evident way as a homogeneous space under $\overline{G}$, the associated group $\overline{H}$ being the reciprocal image of H in $\overline{G}$. If G is effective then $\overline{G}$ is "almost effective," that is, only a finite number of elements in $\overline{G}$ induce the identity mapping of W. It is clear that $\overline{H}$ is maximal in $\overline{G}$ and that $r(\overline{H})=r(\overline{G})$; from the last equality it may be deduced readily that $\overline{H}$ is itself a direct product $H_1 \times H_2 \times \cdots \times H_k$ ($H_i \subset G_i$, $i=1, \cdots, k$). One H_i at least must be different from the G_i in which it lies; let us suppose that $H_1 \neq G_1$, then, H being maximal in G, we have $H_i=G_i$ ($i=2, 3, \cdots, k$).

$G_2 \times G_3 \times \cdots \times G_k$ is now a *connected* subgroup of $\overline{H}$ which is invariant in $\overline{G}$; it must contain only the identity element if $\overline{G}$ is almost effective; therefore, G is isomorphic to G_1/N and is simple, q.e.d.

In [2] D. Montgomery and H. Samelson also determined the simple groups which act transitively on S^n. Their method is of topological nature and requires the knowledge of the homology rings of simple groups; it could not be applied to the exceptional groups.

Another method is suggested by the previous considerations; it consists in finding directly the associated group H. We have seen that if G/H is homeomorphic to S^n (n even) then H is connected, maximal (in the sense of the proof of Theorem II), has the same rank as G and a group $\Phi(H)$ of index two in $\Phi(G)$.

In a paper I wrote with J. de Siebenthal (Lausanne), which will appear in the Comment. Math. Helv.,[8] we study the subgroups of maximal rank of compact Lie groups and we give, for each simple group of the Killing-Cartan classification, a list of all types of connected maximal subgroups having the same rank as the group itself. On the other hand, the orders of the groups Φ may be easily computed: for the simple groups, they are to be found for example in [8], for the others, they are given by the relation $\Phi(G_1 \times G_2)=\Phi(G_1) \times \Phi(G_2)$. By studying that list of maximal subgroups, I found that the index of $\Phi(H)$ in $\Phi(G)$ equals 2 only in the following cases:

(a) D_r in B_r, $r=1, 2, \cdots$;[9]

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[8] A summary is given in a note published in C. R. Acad. Sci. Paris vol. 226 (1948) pp. 1662–1664.

[9] I follow the usual notations: B_r and D_r are the unimodular orthogonal groups of respectively $2r+1$ and $2r$ variables, A_r the unimodular unitary group, C_r the unitary symplectic group of $2r$ variables, G_2, F_4 the exceptional groups of 14 and 52 parameters.

(b) A_2 in G_2.

According to the theorem of Hopf and Samelson, the characteristic of the spaces B_r/D_r and G_2/A_2 is two. But it is well known that these spaces are really homeomorphic to spheres (G_2 is the automorphism-group of the Cayley numbers and acts transitively on the purely imaginary Cayley numbers of norm one, which are in a one-to-one correspondence with the points of S^6). Thus we have proved the following two theorems, the first of which is slightly stronger than the result obtained in [**2**, Theorem II, p. 462].

THEOREM III. *The only compact connected simple Lie group acting transitively on the even-dimensional sphere* S^{2r} *is locally isomorphic to* B_r $(r=1, 2, \cdots)$, *and also, for* $r=3$, *to* G_2.

THEOREM IV. *The even-dimensional spheres are, up to a homeomorphism, the only simply-connected spaces of characteristic two on which compact connected Lie groups act transitively.*

Theorem III gives thus an infinity of simply-connected homogeneous spaces of characteristic 2. This fact occurs only for the prime number 2. More precisely, we can assert the following theorem:

THEOREM V. *For each prime number* $p>2$, *there are only a finite number of simply-connected spaces of characteristic* p *on which compact connected Lie groups act transitively.*[10] *These spaces are homeomorphic to*:

(1) $A_{p-1}/A_{p-2}\times T^1$ (*dimension* $2(p-1)$),

(2) $C_p/C_{p-1}\times C_1$ (*dimension* $4(p-1)$),

and, for $p=3$:

F_4/B_4 (*dimension* 16) *and* $G_2/A_1\times A_1$ (*dimension* 8).

This can be checked with the help of the list of maximal subgroups already cited.

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4. ———, *Sur les invariants intégraux de certains espaces homogènes clos . . .*, Polskie Towarzystwo matematyczne, Krakow, Rocznik vol. 8 (1929) pp. 182–225;

[10] The space G/H is automatically of even dimension if $r(H)=r(G)$, for we have for every compact n-parameter Lie group the relation $n\equiv r(G)$ (modulo 2) (see [**7**, p. 359]).

also *Selecta*, Paris, 1939, pp. 203–233.
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A NOTE ON LEAST COMMON LEFT MULTIPLES

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1. Introduction. Consider n-by-n matrices $A, B, \cdots$ with elements in a principal ideal ring and recall the following definitions. If $A=BC$, then A is a *left multiple* of C and C is a *right divisor* of A. If $A=RD$ and $B=PD$, then D is a *common right divisor* of A and B; if, furthermore, D is a left multiple of every common right divisor of A and B, then D is a *greatest common right divisor* of A and B. If $M=PA=QB$, then M is a *common left multiple* of A and B; if, furthermore, M is a right divisor of every common left multiple of A and B, then M is a *least common left multiple* of A and B. If $FE=I$, where I is the identity matrix, then E is a *unimodular* matrix. If E is unimodular, then EA is a *left associate* of A.

The basic tool in the following constructions is the theorem[1] that any given matrix A is the left associate of a uniquely determined matrix A_1, known as the Hermite canonical triangular form, having zeros above the main diagonal, having elements below the main diagonal in a prescribed residue class modulo the diagonal element above, having each diagonal element in a prescribed system of non-associates, and if a diagonal element is zero, having the corresponding row all zero.

C. C. MacDuffee has presented the following method,[2] due in essence to E. Cahen and A. Chatelet, for finding a greatest common

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[1] C. C. MacDuffee, *Matrices with elements in a principal ideal ring*, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 570–573.

[2] C. C. MacDuffee, loc. cit. p. 573.