ON MATRICES WITH ELEMENTS IN A PRINCIPAL IDEAL RING

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We prove the following theorem.

THEOREM 1. *Let $\mathfrak{D}$ be any commutative principal ideal ring without divisors of zero, and A any matrix with elements in $\mathfrak{D}$ whose characteristic equation factors into linear factors in $\mathfrak{D}$. Then there exists a unimodular matrix T, with elements in $\mathfrak{D}$, such that $T^{-1}AT$ has zeros below the main diagonal.*

This theorem was proved by Leavitt [1][1] for the special case of the ring $\mathfrak{H}$ of all functions of a complex variable holomorphic in, and on the boundary of, a closed bounded region R. His paper contains a proof that this set of functions forms such a ring; and gives an essentially algebraic construction for the transforming matrix T of the ring. Since this construction uses only properties of $\mathfrak{H}$ which are shared by all principal ideal rings [**2**; pp. 168–170] for [**3**; vol. 1, pp. 60–67], it can be carried out in all such rings.

The only changes necessary are those of terminology: "Holomorphic functions" must be replaced by "elements of $\mathfrak{D}$." Two elements are called *associated* if they differ by a unit factor. Since the prime ideals of $\mathfrak{H}$ are the ideals generated by the functions $(z-z_0)$, $z_0 \in R$, one must replace "$\alpha(z)$ has a zero of hth order at z_0" by "$\alpha \equiv 0 \bmod (\mathfrak{P}^h)$," and so on. Substituting a constant z_0 for z corresponds to mapping the ring $\mathfrak{H}$ into its homomorphic image $\mathfrak{H}/(z-z_0)$. Thus since $\mathfrak{D}/\mathfrak{P}$ is always a field or our rings $\mathfrak{D}$, the original arguments hold.

The one portion of the proof which might seem difficult to generalize is the use, in the final construction of the transforming matrix, of the theorem on the existence of a holomorphic function whose expansion at a finite number of points is specified to a finite number of terms. But this is simply the theorem that in any ring with unique factorization into prime ideals, we can find an element satisfying a finite number of simultaneous congruences, provided that the moduli are powers of different prime ideals. This can be proved for all principal ideal rings by the argument used to prove the Chinese remainder theorem in [**4**; p. 12].

It is also possible to give a much simpler proof of our theorem.

Received by the editors February 28, 1948.

[1] The numbers in brackets refer to the bibliography at the close of the paper.

Let $\mathfrak{D}$ be any principal ideal ring, A a matrix satisfying our assumption. Since A can be reduced to its Jordan form J by transformation in the quotient field of $\mathfrak{D}$, we conclude as in §2 of [**1**] that there exists a matrix S, with elements in $\mathfrak{D}$ and nonzero determinant, such that

$$AS = SJ. \tag{1}$$

Since J is in Jordan canonical form, it has elements in $\mathfrak{D}$, and zeros below the main diagonal. Now there always exists a unimodular matrix T such that TS has zeros below the main diagonal ([**2**; p. 228]; it is easy to make the slight necessary modification). Then T^{-1} also has elements in $\mathfrak{D}$, and

$$(TAT^{-1})(TS) = (TS)J.$$

It is easy to verify that the product XY of two matrices in normal form [that is, with zeros below the main diagonal] is again in normal form. Further if Y and XY are in normal form, and none of the diagonal elements of Y are zero, then X is in normal form. Since TS and J are already in normal form, and $|TS| \neq 0$, it follows that TAT^{-1} is in normal form. Clearly the diagonal elements of TAT^{-1} are the roots of $|TAT^{-1}-\lambda I| = 0$, which is the characteristic equation of A.

It should be pointed out that we have not yet defined a canonical form, since we have not described what can be done with the nondiagonal elements, and since two matrices with the same diagonal elements can still be dissimilar. For example

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -1 & 3 & 0 \end{pmatrix},$$

are not similar even though their Jordan forms are identical.

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