THE ASYMPTOTIC DISTRIBUTION OF THE SUM OF A RANDOM NUMBER OF RANDOM VARIABLES

HERBERT ROBBINS

1. Introduction. If a random variable (r. v.) Y is the sum of a large but constant number N of independent components

$$Y = X_1 + \cdots + X_N, \tag{1}$$

then under appropriate conditions on the X_j it follows from the central limit theorem that the distribution of Y will be nearly normal. In many cases of practical importance, however, the number N is itself a r. v., and when this is so the situation is more complex.

We shall consider the case in which the X_j $(j=1, 2, \cdots)$ are independent r. v.'s with the same distribution function (d. f.) $F(x) = P[X_j \leqq x]$, and in which the non-negative integer-valued r. v. N is independent of the X_j. The d. f. of N we shall assume to depend on a parameter λ, so that the d. f. of Y is a function of λ which may have an asymptotic expression as $\lambda \rightarrow \infty$. In the degenerate case in which for any integer λ, N is certain to have the value λ, the problem reduces to the ordinary central limit problem for equi-distributed components.

In the general case the d. f. of N for any λ is determined by the values $\omega_k = P[N=k]$ $(k=0, 1, \cdots)$, where the ω_k are functions of λ such that for all λ,

$$\omega_k \geqq 0, \qquad \sum_0^\infty \omega_k = 1.$$

We shall use Greek letters to denote functions of the parameter λ; in particular we define

$$\begin{aligned} \alpha &= E(N) = \sum_0^\infty \omega_k \cdot k, \\ \beta^2 &= E(N^2) = \sum_0^\infty \omega_k \cdot k^2 \qquad \text{(assumed finite for all } \lambda), \\ \gamma^2 &= \operatorname{Var}(N) = \sum_0^\infty \omega_k \cdot (k-\alpha)^2 = \beta^2 - \alpha^2, \\ \theta(t) &= E(e^{i(N-\alpha)t/\gamma}) = \sum_0^\infty \omega_k \cdot e^{i(k-\alpha)t/\gamma}, \end{aligned} \tag{2}$$

Received by the editors January 12, 1948.

the last being the characteristic function (c. f.) of the normalized r. v.

$$M = (N - \alpha)/\gamma. \tag{3}$$

We shall use Latin letters to denote quantities independent of λ; in particular we define

$$\begin{aligned} a &= E(X_j) = \int x dF(x), \\ b^2 &= E(X_j^2) = \int x^2 dF(x), \\ c^2 &= \operatorname{Var}(X_j) = \int (x-a)^2 dF(x) = b^2 - a^2 \quad (0 < c^2 < \infty), \\ f(t) &= E(e^{iX_j t}) = \int e^{ixt} dF(x). \end{aligned} \tag{4}$$

We then have for the r. v. (1),

$$\begin{aligned} E(Y) &= \sum_0^\infty \omega_k \cdot E(X_1 + \cdots + X_k) = \sum_0^\infty \omega_k \cdot ka = \alpha a, \\ E(Y^2) &= \sum_0^\infty \omega_k \cdot E(X_1 + \cdots + X_k)^2 \\ &= \sum_0^\infty \omega_k \{ kb^2 + k(k-1)a^2 \} = \alpha c^2 + \beta^2 a^2, \\ \sigma^2 &= \operatorname{Var}(Y) = \alpha c^2 + \gamma^2 a^2. \end{aligned} \tag{5}$$

We shall be concerned with the normalized r. v.

$$Z = \frac{Y - E(Y)}{(\operatorname{Var}(Y))^{1/2}} = \frac{(X_1 + \cdots + X_N) - \alpha a}{\sigma}, \tag{6}$$

whose c. f. is

$$\begin{aligned} \phi(t) = E(e^{iZt}) &= \sum_0^\infty \omega_k \cdot E(e^{i[(X_1 + \cdots + X_k) - \alpha a/\sigma]t}), \\ &= \sum_0^\infty \omega_k \cdot e^{-i\alpha a t/\sigma} \cdot f^k\left(\frac{t}{\sigma}\right). \end{aligned} \tag{7}$$

By definition, Z has the limiting d. f. $H(x)$ if whenever x is a continuity point of the d. f. $H(x)$, $\lim_{\lambda\to\infty} P[Z \leqq x] = H(x)$, or, equivalently, setting

$$h(t) = \int e^{ixt} dH(x),$$

if for every t,

$$\lim_{\lambda \to \infty} \phi(t) = h(t). \tag{8}$$

In particular, if (8) holds for $h(t) = e^{-t^2/2}$, then for every x,

$$\lim_{\lambda \to \infty} P[Z \leqq x] = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-u^2/2} du \equiv H_0(x),$$

and Y is said to be asymptotically normal (α a, σ).

2. Some general results.

THEOREM 1. *Let*

$$\delta = \frac{\gamma a}{\sigma} = \left(\frac{\gamma^2 a^2}{\alpha c^2 + \gamma^2 a^2} \right)^{1/2} \qquad (0 \leqq \delta \leqq 1). \tag{9}$$

If, as $\lambda \to \infty$,

$$\sigma^2 \to \infty, \qquad \gamma = o(\sigma^2), \tag{10}$$

then

$$\phi(t) = \theta(\delta t) \cdot e^{-t^2(1-\delta^2)/2} + o(1). \tag{11}$$

PROOF. Since from (10),

$$E\left(\frac{N-\alpha}{\sigma^2}\right) = 0, \qquad E\left(\frac{N-\alpha}{\sigma^2}\right)^2 = \frac{\gamma^2}{\sigma^4} \to 0,$$

it follows that $(N-\alpha)/\sigma^2 \to 0$ in probability as $\lambda \to \infty$. Hence for any $d > 0$,

$$2 \cdot P\left[\left| \frac{N-\alpha}{\sigma^2} \right| > d \right] = o(1) \qquad \text{as } \lambda \to \infty. \tag{12}$$

We now write (7) in the form

$$\phi(t) = \sum_{0}^{\infty} \omega_k \cdot e^{i(k-\alpha)at/\sigma} \cdot \left\{ e^{-iat/\sigma} f\left(\frac{t}{\sigma}\right) \right\}^k, \tag{13}$$

and define

$$\phi_1(t) = \left\{ \sum_{0}^{\infty} \omega_k \cdot e^{i(k-\alpha)at/\sigma} \right\} \cdot \left\{ e^{-iat/\sigma} f\left(\frac{t}{\sigma}\right) \right\}^\alpha; \tag{14}$$

then

$$\phi(t) - \phi_1(t) = \sum_0^\infty \omega_k \cdot e^{i(k-\alpha)at/\sigma}\left[\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^k - \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^\alpha\right],$$

whence

$$(15)\quad \begin{aligned} |\phi(t) - \phi_1(t)| &\leqq \sum_0^\infty \omega_k \left| \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^k - \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^\alpha \right| \\ &= \sum_{|(k-\alpha)/\sigma^2|>d} + \sum_{|(k-\alpha)/\sigma^2|\leqq d} \\ &\leqq 2 \sum_{|(k-\alpha)/\sigma^2|>d} \omega_k + \left| e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right|^\alpha \\ &\quad \sum_{|(k-\alpha)/\sigma^2|\leqq d} \omega_k \left| \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{k-\alpha} - 1\right| \\ &\leqq 2P\left[\left|\frac{N-\alpha}{\sigma^2}\right| > d\right] \\ &\quad + \operatorname*{Max}_{|r|\leqq d} \left| \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\sigma^2 r} - 1\right|. \end{aligned}$$

From (4) we have as $t \to 0$,

$$f(t) = 1 + iat - \frac{b^2t^2}{2} + o(t^2);$$

hence as $\sigma^2 \to \infty$

$$\begin{aligned} e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right) &= \left\{1 - \frac{iat}{\sigma} - \frac{a^2t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right)\right\} \\ &\quad \cdot \left\{1 + \frac{iat}{\sigma} - \frac{b^2t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right)\right\} \\ &= 1 - \frac{c^2t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right), \end{aligned}$$

$$\begin{aligned} \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\sigma^2} &= \left\{1 - \frac{c^2t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right)\right\}^{\sigma^2} = e^{-c^2t^2/2} + o(1) \\ &= \left\{e^{-t^2/2} + o(1)\right\}^{c^2}. \end{aligned}$$

Thus

$$(16)\qquad \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\alpha} = \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\sigma^2\alpha/\sigma}$$
$$= \left\{e^{-t^2/2} + o(1)\right\}^{c^2\alpha/\sigma^2},$$

and

$$(17)\qquad \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\sigma r} = \left\{e^{-t^2/2} + o(1)\right\}^{c^2 r}.$$

Now fix t and $\epsilon > 0$. Choose $d > 0$, until now arbitrary, so that

$$(18)\qquad \left| z - e^{-t^2/2} \right| < d, \qquad |r| < d$$

imply that

$$(19)\qquad \left| z^{c^2 r} - 1 \right| < \frac{\epsilon}{2}\cdot$$

Then choose λ_1 so that $\lambda > \lambda_1$ implies that

$$(20)\qquad 2P\left[\left|\frac{N - \alpha}{\sigma^2}\right| > d\right] < \frac{\epsilon}{2}$$

and that the $o(1)$ in (17) satisfies the inequality

$$(21)\qquad |o(1)| < d.$$

Then it follows from (15) that for $\lambda > \lambda_1$,

$$|\phi(t) - \phi_1(t)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

since ϵ was arbitrary we conclude that

$$(22)\qquad \phi(t) = \phi_1(t) + o(1).$$

We can write $\phi_1(t)$ in the form

$$\phi_1(t) = \left\{\sum_0^{\infty} \omega_k \cdot e^{i(k-\alpha)at/\sigma}\right\}\cdot\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\alpha}$$
$$(23)\qquad = \theta\left\{\frac{\gamma a t}{\sigma}\right\}\cdot\left\{e^{-t^2/2} + o(1)\right\}^{\alpha c^2/\sigma^2} = \theta(\delta t)\cdot\left\{e^{-t^2/2} + o(1)\right\}^{1-\delta^2}$$
$$= \theta(\delta t)\cdot\left\{e^{-t^2/2}(1 + o(1))\right\}^{1-\delta^2}$$
$$= \theta(\delta t)\cdot e^{-t^2(1-\delta^2)/2} + o(1),$$

which, with (22), completes the proof of the theorem.

COROLLARY 1. *If* (10) *holds, and if as* $\lambda \to \infty$,

$$a^2\gamma^2 = o(\alpha), \tag{24}$$

then

$$\lim_{\lambda\to\infty} \phi(t) = e^{-t^2/2}, \tag{25}$$

so that Z has the limiting d. f. $H_0(x)$ *and* Y *is asymptotically normal* $(\alpha\ a,\ \sigma)$.

PROOF. From (24) it follows that as $\lambda\to\infty$, $\delta\to 0$. Moreover, considering the r. v.

$$M_1 = \frac{(N-\alpha)\delta}{\gamma}$$

we have $E(M_1)=0$, $E(M_1^2)=\delta^2\to 0$, so that $M_1\to 0$ in probability. It follows that

$$E(e^{iM_1t}) = \sum_0^\infty \omega_k\cdot e^{i(k-\alpha)\delta t/\gamma} = \theta(\delta t) \to 1, \tag{26}$$

while

$$e^{-t^2(1-\delta^2)/2} \to e^{-t^2/2}, \tag{27}$$

so that (25) follows from (11).

Until now we have not assumed that the normalized r. v. M defined by (3) has a limiting d. f. $G(x)$ as $\lambda\to\infty$.

COROLLARY 2. *If* (10) *holds, and if* N *is asymptotically normal* (α,γ), *then* Z *has the limiting* d. f. $H_0(x)$ *and* Y *is asymptotically normal* $(\alpha a, \sigma)$.

PROOF. In this case we have

$$\lim_{\lambda\to\infty}\theta(\tau) = e^{-\tau^2/2},$$

and the convergence is uniform in the interval $0\leqq\tau\leqq t$. Since $0\leqq\delta\leqq 1$ it follows that as $\lambda\to\infty$,

$$\theta(\delta t) = e^{-(\delta t)^2/2} + o(1), \tag{28}$$

and therefore from (11),

$$\phi(t) = \left\{e^{-\delta^2t^2/2} + o(1)\right\}\cdot\left\{e^{-(1-\delta^2)t^2/2}\right\} + o(1) = e^{-t^2/2} + o(1).$$

The assumption (10) is actually superfluous in this case as we shall see later (Corollary 4).

Let us now consider the case in which M has a non-normal limiting d. f.

COROLLARY 3. *If* (10) *holds, and if M has a non-normal limiting* d. f. $G(x)$, *so that*

$$\lim_{\lambda\to\infty} \theta(t) = g(t) = \int e^{ixt}dG(x) \neq e^{-t^2/2}, \tag{29}$$

and if

$$\lim_{\lambda\to\infty} \frac{c^2\alpha}{a^2\gamma^2} = s \tag{30}$$

exists, $0 \leqq s < \infty$, *then*

$$\lim_{\lambda\to\infty} \phi(t) = g\left(\frac{t}{(1+s)^{1/2}}\right)\cdot e^{-(t(s/(1+s))^{1/2})^2/2} \neq e^{-t^2/2}, \tag{31}$$

so that Z has the non-normal limiting d. f.

$$H(x) = G(t(1+s)^{1/2})_* H_0\left(x\left(\frac{1+s}{s}\right)^{1/2}\right), \tag{32}$$

*where * denotes the operation of convolution.*

PROOF. In this case as $\lambda\to\infty$, $\delta\to(1+s)^{-1/2}$, whence (31) follows as before.

If $s=0$ (that is, if $\alpha=o(a^2\gamma^2)$) then $\lim_{\lambda\to\infty} \phi(t)=\lim_{\lambda\to\infty} \theta(t)=g(t)$, so that Y has the same asymptotic distribution as N. If $0<s<\infty$ then the limiting d. f. of Z is the convolution of a normal with a non-normal d. f. If $s=\infty$ we refer to Corollary 1.

LEMMA 1. *If M has a limiting d. f. $G(x)$ such that $G(x)>0$ for every finite x, then* (10) *holds.*

PROOF. First we shall show that $\gamma=o(\alpha)$ as $\lambda\to\infty$. Suppose not. Then there exists a constant $B>0$ such that for any λ_1 there exists a $\lambda>\lambda_1$ such that

$$\alpha/\gamma < B. \tag{33}$$

We may assume that $-B$ is a continuity point of $G(x)$. Now choose λ_1 so that for all $\lambda>\lambda_1$,

$$P\left[\frac{N-\alpha}{\gamma} \leqq -B\right] > G(-B) - \frac{G(-B)}{2} = \frac{G(-B)}{2} > 0; \tag{34}$$

then for some $\lambda>\lambda_1$ we have both (33) and (34), whence

$$0 = P[N < 0] = P\left[\frac{N-\alpha}{\gamma} < \frac{-\alpha}{\gamma}\right] \geqq P\left[\frac{N-\alpha}{\gamma} \leqq -B\right] > 0,$$

a contradiction. It follows that $\gamma = o(\alpha)$ and hence $\gamma = o(\sigma^2)$.

We shall now show that $\alpha \to \infty$. If not, then since $\gamma = o(\alpha)$, it follows that $\gamma \to 0$, which we shall show to be impossible.

From Tchebychef's inequality, $\gamma \to 0$ implies that

$$P[|N - \alpha| < 1/2] \to 1.$$

But there is at most one integer k satisfying $|k-\alpha| < 1/2$; denoting this integer by k_λ we have

$$P[N = k_\lambda] \to 1.$$

Define

$$L = \liminf_{\lambda\to\infty} \left\{\frac{k_\lambda - \alpha}{\gamma}\right\}.$$

Either $L > -\infty$ or $L = -\infty$. If the former, let $x < L$ be a continuity point of $G(x)$. Then

$$G(x) = \lim_{\lambda\to\infty} P\left[\frac{N-\alpha}{\gamma} \leqq x\right].$$

But for sufficiently large λ,

$$\frac{k_\lambda - \alpha}{\gamma} > x,$$

whence

$$P\left[\frac{N-\alpha}{\gamma} \leqq x\right] < 1 - P[N = k_\lambda].$$

It follows that $G(x) = 0$, a contradiction. On the other hand suppose $L = -\infty$. Then for any x and sufficiently large λ,

$$\frac{k_\lambda - \alpha}{\gamma} < x,$$

whence

$$P\left[\frac{N-\alpha}{\gamma} \leqq x\right] \geqq P[N = k_\lambda].$$

It follows that $G(x) = 1$. Since x was arbitrary, $G(x)$ is not a d. f.

Thus $\alpha\rightarrow\infty$ and hence $\sigma^2\rightarrow\infty$. This completes the proof.

It follows that in Corollary 3 we may drop the assumption (10) provided $G(x)>0$. Moreover, Corollary 2 may now be given its final form.

COROLLARY 4. *If N is asymptotically normal (α, γ) then Y is asymptotically normal $(\alpha a, \sigma)$.*

We shall conclude this section with a theorem concerning the "singular" case in which α and γ are of the same order as $\lambda\rightarrow\infty$, and $a=0$, so that (10) does not hold.

THEOREM 2. *Let $a=0$. If as $\lambda\rightarrow\infty$*

$$\alpha \rightarrow \infty, \qquad \gamma/\alpha \rightarrow r \qquad (0 < r < \infty), \tag{35}$$

and if M has a limiting d. f. $G(x)$ (necessarily such that $G(x)=0$ for some x), then

$$\lim_{\lambda\rightarrow\infty} \phi(t) = \int_0^\infty e^{-t^2y/2}dG_1(y) = g_1\left(\frac{it^2}{2}\right), \tag{36}$$

where

$$G_1(x) = G\left(\frac{x-1}{r}\right), \qquad g_1(t) = \int_0^\infty e^{itx}dG_1(x). \tag{37}$$

Thus the limiting d. f. of Z is

$$H(x) = \int_0^\infty H_0\left(\frac{x}{y^{1/2}}\right)dG_1(y). \tag{38}$$

PROOF. We have for $a=0$,

$$\phi(t) = \sum_0^\infty \omega_k\cdot f^k\left(\frac{t}{(\alpha c^2)^{1/2}}\right) = \sum_0^\infty \omega_k\left\{f^\gamma\left(\frac{t}{(\alpha c^2)^{1/2}}\right)\right\}^{k/\gamma}.$$

Now as $\lambda\rightarrow\infty$,

$$f^\gamma\left(\frac{t}{(\alpha c^2)^{1/2}}\right) = \left\{f^\alpha\left(\frac{t}{(\alpha c^2)^{1/2}}\right)\right\}^{\gamma/\alpha} \rightarrow e^{-rt^2/2}.$$

Set

$$\zeta = \frac{-i\log f^\gamma(t/(\alpha c^2)^{1/2})}{r};$$

then as $\lambda\rightarrow\infty$,

$$\zeta \to it^2/2, \tag{39}$$

and

$$\phi(t) = \sum_0^\infty \omega_k \cdot e^{ir\zeta k/\gamma}. \tag{40}$$

Now let

$$M_1 = rN/\gamma;$$

then for any x such that $(x-1)/r$ is a continuity point of $G(x)$,

$$P[M_1 \leqq x] = P\left[\frac{rN}{\gamma} \leqq x\right] = P\left[\frac{N-\alpha}{\gamma} \leqq \frac{x}{r} - \frac{\alpha}{\gamma}\right]$$
$$\to G\left(\frac{x-1}{r}\right) = G_1(x),$$

where $G_1(x)$ is defined by (37). It follows that

$$E(e^{izM_1}) = \sum_0^\infty \omega_k \cdot e^{izrk/\gamma} \to g_1(z) = \int_0^\infty e^{izy} dG_1(y)$$

uniformly for every z in some neighborhood of $z = it^2/2$. Hence from (39) and (40),

$$\lim_{\lambda\to\infty} \phi(t) = g_1\left(\frac{it^2}{2}\right) = \int_0^\infty e^{-t^2 y/2} dG_1(y).$$

Since

$$\int_0^\infty e^{-t^2 y/2} dG_1(y) = \int_0^\infty \int_{-\infty}^\infty e^{itx} d_x H_0\left(\frac{x}{y^{1/2}}\right) dG_1(y)$$
$$= \int_{-\infty}^\infty e^{itx} d_x \left\{ \int_0^\infty H_0\left(\frac{x}{y^{1/2}}\right) dG_1(y) \right\},$$

it follows that the limiting d. f. of Z is given by (38). This completes the proof of Theorem 2.

From the relation $M_1 = rM + (r\alpha)/\gamma$ it follows that $g_1(t) = e^{it} \cdot g(rt)$, where $g(t)$ is defined by (29). Hence (36) may be written in the equivalent form

$$\lim_{\lambda\to\infty} \phi(t) = e^{-t^2/2} \cdot g\left(\frac{irt^2}{2}\right).$$

3. Some examples. (i) Let N have a Poisson distribution with parameter λ, so that

$$\omega_k = e^{-\lambda}\cdot(\lambda^k/k!) \qquad (k = 0, 1, \cdots);$$

then

$$\alpha = \gamma^2 = \lambda, \qquad \sigma^2 = \lambda b^2.$$

From Corollary 4 it follows that Y is asymptotically normal $(\lambda a, b\,\lambda^{1/2})$. Note that (10) holds but (24) does not.

(ii) Let N have a binomial distribution with parameters λ, p, where λ is an arbitrary positive integer and p and $q=1-p$ are constants, so that

$$\omega_k = \frac{\lambda!}{k!(\lambda - k)!}\, p^k q^{\lambda-k} \qquad (k = 0, 1, \cdots, \lambda);$$

then

$$\alpha = \lambda p, \qquad \gamma^2 = \lambda pq, \qquad \sigma^2 = \lambda p(c^2 + qa^2).$$

Again it follows from Corollary 4 that Y is asymptotically normal $(\lambda pa, (\lambda p(c^2+qa^2))^{1/2})$.

(iii) For any integer λ suppose that N can assume the two values λ, 2λ, with probability 1/2 in each case. Then

$$\alpha = \frac{3\lambda}{2}, \qquad \gamma^2 = \frac{\lambda^2}{4}, \qquad \sigma^2 = \frac{3\lambda}{2}c^2 + \frac{\lambda^2}{4}a^2.$$

First suppose $a\neq 0$. Then as $\lambda\to\infty$ (10) holds, and the quantity s of Corollary 3 is 0. Moreover, $\theta(t)=\cos t$, so that M has the non-normal limiting distribution for which $P[M=-1]=P[M=1]=1/2$. It follows from Corollary 3 that Z has the same limiting distribution.

The case is quite different when $a=0$, for then $\gamma\neq o(\sigma^2)$, and Theorem 2 applies. We have

$$r = \lim_{\lambda\to\infty}\frac{\gamma}{\alpha} = \frac{1}{3}, \qquad g_1(t) = \frac{1}{2}\left\{e^{2it/3} + e^{4it/3}\right\},$$

so that

$$\lim_{\lambda\to\infty}\phi(t) = \frac{1}{2}\left\{e^{-t^2/3} + e^{-2t^2/3}\right\}.$$

Thus the limiting d. f. of Z is a mixture of two normal d. f.'s with means 0 and variances 2/3 and 4/3.

UNIVERSITY OF NORTH CAROLINA