# ON THE HOMOTOPY TYPE OF ANR'S

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**1. Introduction.** If $X$ and $Y$ are any spaces and if $f: X \to Y$ and $g: Y \to X$ are maps such that $gf \simeq 1$, then $g$ is called a *left homotopy inverse* of $f$ and $f$ a *right homotopy inverse*[1] of $g$. In this case we shall say that $Y$ *dominates*[2] $X$. If $Y$ dominates $X$ and $Z$ dominates $Y$ then it is easily verified that $Z$ dominates $X$. If $g$ is both a right and left homotopy inverse of $f$ it is called a *homotopy inverse* of $f$ and $f$ will be called a *homotopy equivalence*. Thus the assertion that $f: X \to Y$ is a homotopy equivalence claims that $X$ and $Y$ are of the same homotopy type and, moreover, that $f$ has a homotopy inverse.

Two maps, $f_0, f_1: X \to Y$ are said (cf. [**1**, pp. 49, 50] and [**2**, p. 344]) to be $n$-homotopic if, and only if, $f_0\phi \simeq f_1\phi$ for every map, $\phi: P \to X$, of every (finite) polyhedron, $P$, of at most $n$ dimensions. By an *n-homotopy inverse* of a map, $f: X \to Y$, or an *n-homotopy equivalence* we mean the same as a homotopy inverse or a homotopy equivalence with homotopy replaced by $n$-homotopy throughout the definition.

By a CR-space we shall mean a connected compactum, which is an ANR (absolute neighborhood retract). Any CR-space, $X$, is dominated by a finite simplicial complex [**5**, Theorems 12.2, 16.2, pp. 93, 99], even if its dimensionality is infinite. We shall use $\Delta X$ to denote the minimum dimensionality of all (finite, simplicial) complexes which dominate $X$. Then $\Delta X \leqq \dim X$ and we may think of $\Delta X$ as a kind of "quasi-dimensionality," noticing, however, that $\Delta X$ may be less than $\dim X$, even if $X$ is itself a finite polyhedron.

Let $X$, $Y$ be CR-spaces, and let $N = \max(\Delta X, \Delta Y)$. Let $f: X \to Y$ be a given map and let $f_n: \pi_n(X) \to \pi_n(Y)$ be the homomorphism induced by $f$. If $f$ is a homotopy equivalence then $f_n$ is an isomorphism onto for each $n \geqq 1$. In §3 below we prove a sharper theorem than the converse, namely:

THEOREM 1. *If $f_n: \pi_n(X) \to \pi_n(Y)$ is an isomorphism onto for each $n = 1, \cdots, N$, then $f: X \to Y$ is a homotopy equivalence.*[3]

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[1] Cf. [**1**]. Numbers in brackets refer to the references cited at the end of the paper.

[2] In this case the homomorphisms $H_n(Y) \to H_n(X)$ induced by $g: Y \to X$ are all onto, likewise the induced homomorphisms $\pi_n(Y) \to \pi_n(X)$, assuming $X$, $Y$ to be arcwise connected. In fact $H_n(Y)$, or $\pi_n(Y)$ $(n \geqq 2)$, may be represented as the direct sum of $H_n(X)$, or $\pi_n(X)$, and the kernel of this homomorphism.

[3] If $X$ and $Y$ are of the same homotopy type, then each dominates the other and $\Delta X = \Delta Y$. Theorem 1 is formulated with a view to applications in which it is possible to calculate separate upper bounds for $\Delta X$, $\Delta Y$ (for example, $\dim X$, $\dim Y$).

We shall also prove:

THEOREM 2. *The map $f:X\rightarrow Y$ is an $(N-1)$-homotopy equivalence if, and only if, $f_n:\pi_n(X)\rightarrow\pi_n(Y)$ is an isomorphism onto for each $n=1, \cdots, N-1$.*

If $\Delta X=\Delta Y=0$ then it is obvious that $X$ and $Y$, being connected, are both absolute retracts. Therefore any map, $X\rightarrow Y$, is a homotopy equivalence and Theorem 1, likewise Theorem 3 below, is trivial. Similarly Theorems 2 and 4 are trivial if $N\leqq 1$. Therefore we shall assume that $N\geqq 1$ in Theorems 1 and 3 and $N\geqq 2$ in Theorems 2 and 4.

Theorem 2 is significant in the theory of polyhedra or cell complexes. For the $(n-1)$-homotopy type of the $n$-section is a homotopy invariant of a given complex $K$ (that is, is the same for any complex of the same homotopy type). It is equivalent to what I have previously called the $n$-group (see [**6**] and [**7**]) of $K$, but now propose to call the $n$-type. These statements will be proved in a later paper in which the $n$-type of a complex will be further discussed.

A map $f:X\rightarrow Y$ is not necessarily an $m$-homotopy equivalence if $f_1, \cdots, f_m$ are isomorphisms onto, where $m<N-1$. For example, let $Y$ be a complex projective plane, let $X\subset Y$ be a 2-sphere, which is a complex line in $Y$, and let $f:X\rightarrow Y$ be the identity. If $g: Y\rightarrow X$ were a 2-homotopy inverse of $f$, then $g|X=gf:X\rightarrow X$ would be of degree $+1$ and would therefore induce the identical automorphism of $\pi_3(X)$. But $f_3\pi_3(X)=0$. Therefore it would be absurd to suppose that $g|X$ can be extended to a map $g: Y\rightarrow X$.

Theorem 1, restricted to polyhedra and weakened by replacing $N$ by max (dim $X+1$, dim $Y$), is essentially a restatement of parts of Theorems 15 and 17 in [**6**, pp. 273 and 277]. The generalization to CR-spaces was suggested by a theorem proved by Sze-Tsen Hu in [**9**]. However we do not actually use Hu's theorem. Instead we follow Lefschetz's approach to the subject and eventually deduce Hu's theorem in a modified form. Of course Theorem 1 does not mean that $X$ and $Y$ are necessarily of the same homotopy type if $\pi_n(X)\approx\pi_n(Y)$ for all values[4] of $n$. The crux of the matter is not merely that $\pi_n(X)\approx\pi_n(Y)$ but that a certain set of isomorphisms, $f_n:\pi_n(X)\rightarrow\pi_n(Y)$, can be "realized geometrically" by means of a map $f:X\rightarrow Y$.

Let $\tilde{X}$ be the universal covering space of a given CR-space, $X$, with base point $x_0\in X$. Then a point, $\tilde{x}\in\tilde{X}$, is a homotopy class of

[4] For example $\pi_n(P^4\times S^3)\approx\pi_n(S^5\times S^2)$ for every $n\geqq 1$, where $P^4$ is a complex projective plane and $S^r$ is an $r$-sphere. This example is due to Hsien-Chung Wang. For other examples see [**10**].

paths, with fixed end points, joining $x_0$ to the point $p\tilde{x}\in X$, where $p$ is the projection, $p:\tilde{X}\to X$, which is thus defined. Thus $\tilde{X}$ contains a base point, $\tilde{x}_0\in\tilde{X}$, which corresponds to the constant path on $x_0$. If $\pi_1(X)=1$ we identify $\tilde{X}$ with $X$, taking $\tilde{x}=p\tilde{x}$. Let $H_n(\tilde{X})$ be the $n$th homology group[5] of $\tilde{X}$. Let $Y$ be another CR-space, let $\tilde{Y}$, $H_n(\tilde{Y})$ be similarly defined and let $p$ also denote the projection $p:\tilde{Y}\to Y$. Then a given map $f:X\to Y$ can be "lifted" into a unique map $\tilde{f}:\tilde{X}\to\tilde{Y}$, such that $fp=p\tilde{f}$, $\tilde{f}\tilde{x}_0=\tilde{y}_0$, where $y_0=fx_0$ and $\tilde{y}_0$ are the base points in $Y$ and $\tilde{Y}$. The map $\tilde{f}$ induces homomorphisms $H_n(\tilde{X})\to H_n(\tilde{Y})$, which we shall also describe as *induced* by $f:X\to Y$. In particular $Y$ may be a finite, $\Delta X$-dimensional polyhedron, which dominates $X$, and $f:X\to Y$ a map with a left inverse $g: Y\to X$. Let $\tilde{g}:\tilde{Y}\to\tilde{X}$ be the map obtained by lifting $g$. Then a homotopy $gf\to 1$ may be lifted into a homotopy $\tilde{g}\tilde{f}\to u$, where $u:\tilde{X}\to\tilde{X}$ is a transformation in the covering group (that is, $pu=p$). Therefore $u^{-1}\tilde{g}$ is a left homotopy inverse of $\tilde{f}$ (likewise $\tilde{f}u^{-1}$ is a right homotopy inverse of $\tilde{g}$) and $\tilde{Y}$ dominates $\tilde{X}$. It follows that $H_n(\tilde{X})=0$ if $n>\dim\tilde{Y}=\dim Y=\Delta X$. Therefore, if $X$, $Y$ are any two CR-spaces, $H_n(\tilde{X})=0$, $H_n(\tilde{Y})=0$ if $n>\max(\Delta X, \Delta Y)$. We shall prove:

THEOREM 3. *If $X$, $Y$ are any CR-spaces, then a map $f:X\to Y$ is a homotopy equivalence if each of the induced homomorphisms $f_1:\pi_1(X)\to\pi_1(Y)$, $H_n(\tilde{X})\to H_n(\tilde{Y})$ $(n=2, 3, \cdots)$ is an isomorphism onto.*

As a corollary to this we have:

COROLLARY 1. *If $X$, $Y$ are simply connected CR-spaces, then a map $f:X\to Y$ is a homotopy equivalence if each of the induced homomorphisms $H_n(X)\to H_n(Y)$ $(n=2, 3, \cdots)$ is an isomorphism onto.*

Let $X$ be a finite cell complex.[6] Then the groups $H_n(\tilde{X})$ may be defined in terms of chain groups,[7] $C_n(\tilde{X})$, which are free $\mathfrak{R}(X)$-modules, where $\mathfrak{R}(X)$ is the group ring of $\pi_1(X)$, with integral coefficients. According to Eilenberg and Steenrod a map, $f:X\to Y$, of $X$ into another complex $Y$, is said to be *cellular* if, and only if, $fX^n\subset Y^n$ for each $n=0, 1, \cdots$, where $X^n$, $Y^n$ are the $n$-sections of

[5] It is to be understood that all our homology groups are defined, as in [**12**], in terms of singular chains with integral coefficients.

[6] That is, a complex of the sort defined on p. 1235 of [**7**] or in a forthcoming book by S. Eilenberg and N. E. Steenrod.

[7] Cf. [**11**, chap IV, §17]. The generalization from ordinary polyhedral complexes to the more general cell complexes will be described in the book by Eilenberg and Steenrod. $C_n(\tilde{X})$ is the relative homology group $H_n(\tilde{X}^n, \tilde{X}^{n-1})$, where $\tilde{X}^r$ is the $r$-section of $\tilde{X}$. For an account of chain mappings and chain equivalences see [**4**] and [**12**].

$X$, $Y$. A cellular map, $f:X\to Y$, determines a chain mapping, $\gamma:C_n(\tilde{X})\to C_n(\tilde{Y})$, which is an operator homomorphism, for each $n=0, 1, \cdots$, in the sense that $\gamma(\rho c)=(\alpha\rho)\gamma c$, where $\rho\in\Re(X)$, $c\in C_n(\tilde{X})$ and $\alpha:\Re(X)\to\Re(Y)$ is the homomorphism induced by $f_1:\pi_1(X)\to\pi_1(Y)$.

A chain mapping $\gamma:C(\tilde{X})\to C(\tilde{Y})$ of the family $C(\tilde{X})=\{C_n(\tilde{X})\}$ into the family $C(\tilde{Y})=\{C_n(\tilde{Y})\}$ is defined in purely algebraical terms as a homomorphism, $\alpha:\pi_1(X)\to\pi_1(Y)$, together with a family of operator homomorphisms, $\gamma:C_n(\tilde{X})\to C_n(\tilde{Y})$, such that $\partial\gamma=\gamma\partial$, where $\partial$ is the boundary operator. If $\gamma$ is the chain mapping, which is induced by some (cellular) map $f:X\to Y$, then $f$ will be described as a *geometrical realization* of $\gamma$. From Theorem 2 we have the corollary:

COROLLARY 2. *If a given chain equivalence*[8] $\gamma:C(\tilde{X})\to C(\tilde{Y})$ *has a geometrical realization,* $f:X\to Y$, *then* $f$ *is a homotopy equivalence.*

This corollary shows that the problem of determining conditions for a given chain mapping to have a geometrical realization is fundamental in the homotopy theory of complexes. In a later paper we shall prove that, if $X$ is at most 3-dimensional, then any chain mapping, $\gamma:C(\tilde{X})\to C(\tilde{Y})$, has a geometrical realization, subject to certain conditions on $\gamma:C_0(\tilde{X})\to C_0(\tilde{Y})$.

Let $X$, $Y$, $f:X\to Y$ be as in Theorem 2 and let $N=\max(\Delta X, \Delta Y)$. Then we prove, as a companion to Theorem 2:

THEOREM 4. *The map* $f:X\to Y$ *is an* $(N-1)$*-homotopy equivalence if*

(a) *each of the induced homomorphisms* $f_1:\pi_1(X)\to\pi_1(Y)$, $H_n(\tilde{X})\to H_n(\tilde{Y})$ $(n=0, \cdots, N-2)$ *is an isomorphism onto,*

(b) *the induced homomorphism* $H_{N-1}(\tilde{X})\to H_{N-1}(\tilde{Y})$ *is onto,*

(c) $f_{N-1}:\pi_{N-1}(X)\to\pi_{N-1}(Y)$ *is an isomorphism into.*

*Conversely, if* $f:X\to Y$ *is an* $(N-1)$*-homotopy equivalence, so is the lifted map,* $\tilde{f}:\tilde{X}\to\tilde{Y}$, *and* $H_n(\tilde{X})\to H_n(\tilde{Y})$ *is an isomorphism onto for* $n=0, 1, \cdots, N-1$.

2. **A lemma on mapping cylinders.** Let $A$, $B$ be any two spaces and $A_0\subset A$, $B_0\subset B$ any subsets of $A$, $B$. We shall say that *the pair* $(B, B_0)$ *dominates* $(A, A_0)$ if, and only if, there are maps, $f:(A, A_0)\to(B, B_0)$ and $g:(B, B_0)\to(A, A_0)$, such that $gf$ is deformable into the identity by a homotopy of the form $\xi_t:(A, A_0)\to(A, A_0)$.

Let $X$, $Y$ be any compacta and $f:X\to Y$ a given map. We form the topological product $X\times I$ and, replacing $X$ by a homeomorph, if necessary, assume that no two of $X$, $Y$, $X\times I$ have a point in common.

[8] It is to be understood that the homomorphism $f_1:\pi_1(X)\to\pi_1(Y)$ associated with a chain equivalence is an automorphism onto.

Let $Z$ be the mapping cylinder, which is formed by identifying[9] $(x, 0)\in X\times I$ with $x$ and $(x, 1)$ with $fx\in Y$ for each $x\in X$. Let $P$, $Q$ be compacta, which dominate $X$, $Y$, respectively, and let $\lambda: X\to P$, $\lambda': P\to X$, $\mu: Y\to Q$, $\mu': Q\to Y$ be maps such that $\lambda'\lambda\simeq 1$, $\mu'\mu\simeq 1$. Let $R$ be the mapping cylinder of the map $\mu f\lambda': P\to Q$. Then our lemma is:

LEMMA 1. *The pair* $(R, P)$ *dominates* $(Z, X)$.

Let $\xi_t: X\to X$ and $\eta_t: Y\to Y$ be homotopies such that $\xi_0=\lambda'\lambda$, $\xi_1=1$, $\eta_0=\mu'\mu$, $\eta_1=1$ and let $\nu: (Z, X)\to(R, P)$ be given by

$$\begin{aligned} \nu(x, t) &= (\lambda x, 2t) && (\text{if } 0 \leqq 2t \leqq 1)\\ &= \mu f\xi_{2t-1}x && (\text{if } 1 \leqq 2t \leqq 2),\\ \nu y &= \mu y && (x \in X,\ y \in Y). \end{aligned}$$

This is single-valued, hence continuous (see [8, §5]), since $(\lambda x, 1) = (\mu f\lambda')\lambda x = \mu f\xi_0 x$, $\mu f\xi_1 x = \mu f x$. Let $\nu': (R, P)\to(Z, X)$ be given by

$$\begin{aligned} \nu'(p, t) &= (\lambda' p, 2t) && (\text{if } 0 \leqq 2t \leqq 1)\\ &= \eta_{2-2t}f\lambda' p && (\text{if } 1 \leqq 2t \leqq 2),\\ \nu' q &= \mu' q && (p \in P,\ q \in Q). \end{aligned}$$

This is single-valued since $(\lambda' p, 1) = f\lambda' p = \eta_1 f\lambda' p$ and $\eta_0 f\lambda' p = \mu' \cdot(\mu f\lambda')p$. The map $\nu'\nu: (Z, X)\to(Z, X)$ is given by

$$\begin{aligned} \nu'\nu(x, t) &= \nu'(\lambda x, 2t) && (\text{if } 0 \leqq 2t \leqq 1)\\ &= \nu'\mu f\xi_{2t-1}x && (\text{if } 1 \leqq 2t \leqq 2),\\ \nu'\nu y &= \nu'\mu y, \end{aligned}$$

or by

$$\begin{aligned} \nu'\nu(x, t) &= (\lambda'\lambda x, 4t) && (\text{if } 0 \leqq 4t \leqq 1)\\ &= \eta_{2-4t}f\lambda'\lambda x && (\text{if } 1 \leqq 4t \leqq 2)\\ &= \mu'\mu f\xi_{2t-1}x && (\text{if } 1 \leqq 2t \leqq 1),\\ \nu'\nu y &= \mu'\mu y. \end{aligned}$$

The desired homotopy, $\zeta_s: (Z, X)\to(Z, X)$, is given by

$$\begin{aligned} \zeta_s(x, t) &= (\xi_s x, (4 - 3s)t) && (\text{if } 0 \leqq t \leqq 1/(4 - 3s))\\ &= \eta_{2-(4-3s)t}f\xi_s x && (\text{if } 1/(4 - 3s) \leqq t \leqq (2 - s)/(4 - 3s))\\ &= \eta_s f\xi_{\rho(s,t)}x && (\text{if } (2 - s)/(4 - 3s) \leqq t \leqq 1),\\ \zeta_s y &= \eta_s y, \end{aligned}$$

[9] The points in $X$, $Y$ shall retain their individualities in $Z$, so that $X$, $Y\subset Z$.

where $\rho(s, t) = \{(4-3s)t+3s-2\}/2$. It is easy to verify that $\zeta_s$ is single-valued and that $\zeta_0 = \nu'\nu$, $\zeta_1 = 1$. Moreover $\zeta_s x = \xi_s x \in X$ if $x = (x, 0) \in X$. Therefore $(R, P)$ dominates $(Z, X)$.

3. **Proof of Theorem** 1. Let $X$, $Y$ and $f: X \to Y$ satisfy the conditions of Theorem 1 and let $Z$ be the mapping cylinder of $f$. The theorem will follow from [**1**, Theorem 3.7, p. 45] (see also [**3**]), when we have proved that $X$ is a deformation retract of $Z$.

Let $g_s: Z \to Z$ be the deformation which is given by $g_s | Y = 1$, $g_s(x, t) = (x, s+t-st)$ $(0 \leqq s \leqq 1)$. Then $g_0 = 1$, $g_1 Z = Y$ and $g_1 x = fx$ for each $x = (x, 0) \in X$. Let $k: Z \to Y$ be the map which is given[10] by $kz = g_1 z$ for each $z \in Z$. Then $g_1 = jk$, $f = ki$, where $i$, $j$ are the identical maps $i: X \to Z$, $j: Y \to Z$. Let $\pi_n(X)$, $\pi_n(Z)$, $\pi_n(Z, X)$ be referred to a base point $x_0 \in X$, and $\pi_n(Y)$ to $fx_0 \in Y$ as base point. Let $i_n: \pi_n(X) \to \pi_n(Z)$, $k_n: \pi_n(Z) \to \pi_n(Y)$ be the homomorphisms induced by $i$, $k$ and let $j_n: \pi_n(Y) \to \pi_n(Z)$ be the homomorphism induced by $j$ and the segment (cf. [**13**] and [**6**, pp. 279 et seq.]) $(x_0, I)$, which joins $x_0$ to $fx_0$. Since $kj = 1: Y \to Y$, $jk = g_1 \simeq 1: Z \to Z$ and since $g_s x_0$ travels along the segment $(x_0, I)$ in the homotopy $g_s$, it follows that $j_n$ is an isomorphism onto and that $k_n$ is its inverse. Since $f = ki$ we have $f_n = k_n i_n$. Therefore $i_n = j_n f_n$ and $i_n$, like $f_n$, is an isomorphism onto for $n = 1, \cdots, N$.

Let $2 \leqq n \leqq N$ and consider the homotopy sequence

$$(3.1) \qquad \pi_n(X) \xrightarrow{1} \pi_n(Z) \xrightarrow{2} \pi_n(Z, X) \xrightarrow{3} \pi_{n-1}(X) \xrightarrow{4} \pi_{n-1}(Z),$$

in which (1) is $i_n$ and (4) is $i_{n-1}$. Since (1) is onto it follows from the exactness of the sequence that (2) maps $\pi_n(Z)$ into zero. Since (4) is an isomorphism it follows that (3) is into zero and (2) is onto. Therefore $\pi_n(Z, X) = 0$ for $n = 1, \cdots, N$, where $\pi_1(Z, X) = 0$ means that[11] $i_1: \pi_1(X) \to \pi_1(Z)$ is onto. Notice that if, in addition, $i_{N+1}$ is onto, then it follows from (3.1) that $\pi_{N+1}(Z, X) = 0$.

Let $P$ be a finite, $\Delta X$-dimensional simplicial complex, which dominates $X$, and $Q$ a finite, $\Delta Y$-dimensional simplicial complex which dominates $Y$. Let $R$, $\lambda$, $\mu$, $\eta_t$, etc. mean the same as in §2. Since $Z$ is (obviously) arcwise connected and since $\pi_n(Z, X) = 0$ for $n = 1, \cdots, N \geqq \dim Q$ it follows from a standard argument[12] that

[10] We distinguish between maps $u: A \to B$, $v: A \to C$, where $B \subset C$, $B \neq C$, even if $ua = va$ for each $a \in A$.

[11] Since $X$ is arcwise connected this is equivalent to the condition that any arc in $Z$, with its end points in $X$, is deformable, with its end points held fixed, into an arc in $X$.

[12] [**14**, p. 526]. This argument is recapitulated, in a slightly more general form, in §8 below (Lemma 5).

there is a homotopy, $\delta_t: Q \to Z$, such that $\delta_0 = \nu' | Q = j\mu'$, $\delta_1 Q \subset X$. Therefore $\delta_t \mu: Y \to Z$ is a homotopy such that $\delta_0 \mu = j\mu'\mu = j\eta_0$, $\delta_1 \mu Y \subset \delta_1 Q \subset X$. Moreover we assume, as we obviously may, that $\delta_1 \mu f x_0 = x_0$, where $x_0$ is the base point of $\pi_n(X)$ and $\pi_n(Z)$. Let $h_t: Y \to Z$ be the homotopy which is given by

$$\begin{aligned} h_t &= j\eta_{1-2t} && (\text{if } 0 \leqq 2t \leqq 1) \\ &= \delta_{2t-1}\mu && (\text{if } 1 \leqq 2t \leqq 2). \end{aligned}$$

Then $h_0 = j$, $h_1 Y \subset X$, $h_1 x_0 = x_0$. Therefore the resultant of the homotopy $g_t: Z \to Z$, followed by $h_t k: Z \to Z$, is a deformation, $\Delta_t: Z \to Z$, such that $\Delta_0 = 1$, $\Delta_1 Z \subset X$, $\Delta_1 x_0 = x_0$.

Let $c \in \pi_1(Z)$ be the element which is represented by the track of $x_0$ in the homotopy $\Delta_t$. Since $i_1$ is onto we have $c = i_1 a$ for some $a \in \pi_1(X)$. Let $\theta_0: (S^n, p_0) \to (Z, x_0)$ be a map representing a given element $\gamma \in \pi_n(Z) (n \geqq 2)$, where $p_0$ is the base point in the standard $n$-sphere $S^n$. Then $\theta_t = \Delta_t \theta_0$ is a deformation of $\theta_0$ into the map $\theta_1 = i\theta$, where $\theta: (S^n, p_0) \to (X, x_0)$ is given by $\theta p = \Delta_1 \theta_0 p$ $(p \in S^n)$. Therefore

$$\gamma = c i_n \alpha = (i_1 a)(i_n \alpha) = i_n(a\alpha),$$

where $\alpha \in \pi_n(X)$ is the element represented by $\theta$ and $ci_n\alpha$, $a\alpha$ are the images of $i_n\alpha$, $\alpha$ in the automorphisms ([**13**] and [**6**, pp. 279 et seq.]) determined by $c$, $a$. Hence $i_n$ is onto for every $n \geqq 1$. Taking $n = N+1$, it follows from (3.1) that $\pi_{N+1}(Z, X) = 0$.

It follows from an extension of a theorem due to Borsuk ([**15**] and [**8**]) that $Z$ is an ANR. Therefore the homotopy, $\delta_t: Q \to Z$, defined above, can be extended, first throughout $P \cup Q$ by defining $\delta_t p = \nu' p \in X$ if $p \in P$, and then to a homotopy $\delta_t': R \to Z$, such that $\delta_0' = \nu'$. Since $\pi_n(Z, X) = 0$ if $1 \leqq n \leqq N+1$ and since $N+1 \geqq \dim R$, it follows from repeated applications of Lemma 5, in §8 below (cf. the proof of Lemma 6), that there is a homotopy, $\rho_t: R \to Z$, rel. $P$ (that is, $\rho_t | P = \rho_0 | P$), such that $\rho_0 = \delta_1'$, $\rho_1 R \subset X$. The resultant of $\delta_t'$, followed by $\rho_t$, is a homotopy, $\phi_t: R \to Z$, rel. $P$, such that $\phi_0 = \nu'$, $\phi_1 R \subset X$. Then $\phi_t \nu: Z \to Z$ is a homotopy such that $\phi_0 \nu = \nu' \nu = \zeta_0$, $\phi_1 \nu Z \subset \phi_1 R \subset X$. Therefore the resultant of $\zeta_{1-t}: (Z, X) \to (Z, X)$, followed by $\phi_t \nu$, is a homotopy, $\psi_t: (Z, X) \to (Z, X)$, such that $\psi_0 = 1$, $\psi_1 Z \subset X$. It follows from [**1**, Theorem 1.4, p. 42], and [**3**] that $X$ is a deformation retract of $Z$ and the proof is complete.

Notice that Theorem 1 follows more directly on the alternative hypothesis that $f_n: \pi_n(X) \to \pi_n(Y)$ is an isomorphism onto for $n = 1, \cdots, m-1$ and $f_m$ is onto, where $m = \max(\Delta X + 1, \Delta Y) = \dim R$. For in this case it follows from (3.1) that $\pi_n(Z, X) = 0$ for $n = 1, \cdots, m$ and the paragraph showing that $\pi_{N+1}(Z, X) = 0$ is unnecessary.

Let $Z$ and $X \subset Z$ be CR-spaces and let $\pi_n(Z, X)=0$ for $n=1, \cdots, m=\max(\Delta X+1, \Delta Z)$. Then it follows from the homotopy sequence (3.1) that $i_n: \pi_n(X) \to \pi_n(Z)$ is an isomorphism onto for $n=1, \cdots, m-1$ and that $i_m$ is onto. Therefore the identity map $i: X \to Z$ is a homotopy equivalence, whence $X$ is a deformation retract of $Z$. Hence we have the modified form of Hu's generalization of Hurewicz's theorem ([**14**, Theorem IV, p. 522] and [**9**]):

COROLLARY.[13] *If* $\pi_n(Z, X)=0$ *for* $n=1, \cdots, \max(\Delta X+1, \Delta Z)$, *then* $X$ *is a deformation retract of* $Z$.

4. **Proof of Theorem 2.** Let $f_n: \pi_n(X) \to \pi_n(Y)$ be an isomorphism onto for $n=1, \cdots, N-1$. Using the same notation as in §3, we shall prove that $i: X \to Z$ is an $(N-1)$-homotopy equivalence. Since $k: Z \to Y$ is a homotopy equivalence and $f=ki$ it will then follow that $f$ is an $(N-1)$-homotopy equivalence. It follows from (3.1) that $\pi_n(Z, X)=0$ for $n=1, \cdots, N-1$. Therefore there is a homotopy, $\delta_t: Q \to Z$, such that $\delta_0=j\mu'$, $\delta_1 Q^{N-1} \subset X$, where $Q^n$ is the $n$-section of $Q$. Since $i_{N-1}$ is an isomorphism it follows from an argument which is similar to one used in proving Lemma 6, in §8 below, that $\delta_1 | Q^{N-1}$ can be extended to a map $i\theta$, where $\theta$ is of the form $\theta: Q \to X$. Then $i\theta | Q^{N-1} \simeq j\mu' | Q^{N-1}$. Since $\theta$, $\mu$, $k$ are of the form $k: Z \to Y$, $\mu: Y \to Q$, $\theta: Q \to X$ they have a resultant $\theta\mu k: Z \to X$. I say that $\theta\mu k$ is an $(N-1)$-homotopy inverse of $i: X \to Z$. For let $K$ be a finite polyhedron of at most $N-1$ dimensions and let $\phi: K \to X$ be a given map. Then $\mu k i\phi$ maps $K$ into $Q$ and is homotopic to a map, $\phi': K \to Q$, such that $\phi' K \subset Q^{N-1}$. Since $i\theta | Q^{N-1} \simeq j\mu' | Q^{N-1}$ it follows that $i\theta\phi' \simeq j\mu'\phi'$. Therefore

$$i\theta\mu k i\phi \simeq i\theta\phi' \simeq j\mu'\phi' \simeq j\mu'\mu k i\phi.$$

Since $\mu'\mu \simeq 1$, $jk \simeq 1$ it follows that

$$i\theta\mu k i\phi \simeq jki\phi \simeq i\phi$$

and hence that $\theta\mu k i\phi \simeq \phi$, according to Lemma 6 below. Therefore $\theta\mu k i \simeq_{N-1} 1$. A similar but rather simpler argument shows that $i\theta\mu k \simeq_{N-1} 1$. Therefore $i$, and hence $f: X \to Y$, is an $(N-1)$-homotopy equivalence.

Conversely[14] let $f: X \to Y$ have an $(N-1)$-homotopy inverse,

[13] If $\dim(Z-X) < \Delta X+1$ this is weaker than Hu's version.

[14] This is nontrivial because of questions concerning the base point. For example, let $u, u': (S^n, p_0) \to (X, x_0)$ be maps representing two given elements of $\pi_n(X) = \pi_n(X, x_0)$. Assume that $gfx_0=x_0$. Then the images of $p_0$ will, in general, describe circuits, which represent different elements of $\pi_1(X, x_0)$, in the homotopies $gfu \simeq u$, $gfu' \simeq u'$.

$g: Y \to X$. Let vertices $p_0 \in P^0$, $q_0 \in Q^0$ be chosen as base points for all the groups $\pi_n(P)$, $\pi_n(Q)$ and let $x_0 = \lambda' p_0$, $y_0 = \mu' q_0$ be taken as base points for $\pi_n(X)$, $\pi_n(Y)$. After suitable homotopies of $\lambda$, $\mu$, if necessary, we assume that $\lambda x_0 = p_0$, $\mu y_0 = q_0$. We then deform $f$ so that $f x_0 = y_0$ and finally $g$ so that $g y_0 = x_0$. Let $g_n: \pi_n(Y) \to \pi_n(X)$ be the homomorphism induced by $g$ and let $\lambda_n$, $\mu_n$, $\lambda_n'$, $\mu_n'$ be the homomorphisms of $\pi_n(X)$, $\pi_n(Y)$, $\pi_n(P)$, $\pi_n(Q)$, which are induced by $\lambda$, $\mu$, $\lambda'$, $\mu'$. Since $\lambda'\lambda \simeq 1$, $\mu'\mu \simeq 1$ we have $\lambda_n'\lambda_n = \alpha_n$, $\mu_n'\mu_n = \beta_n$, where $\alpha_n$, $\beta_n$ are automorphisms[15] of $\pi_n(X)$, $\pi_n(Y)$. Also $gf\lambda' | P^{N-1} \simeq \lambda' | P^{N-1}$, whence $g_n f_n \lambda_n' = \alpha_n' \lambda_n'$ if $1 \leqq n \leqq N-1$, where $\alpha_n' : \pi_n(X) \to \pi_n(X)$ is an automorphism. Therefore

$$\begin{aligned} g_n f_n \alpha_n &= g_n f_n \lambda_n' \lambda_n = \alpha_n' \lambda_n' \lambda_n \\ &= \alpha_n' \alpha_n, \end{aligned}$$

whence $g_n f_n = \alpha_n'$. Similarly $f_n g_n = \beta_n'$ where $\beta_n' : \pi_n(Y) \to \pi_n(Y)$ is an automorphism. Hence[16] it easily follows that $f_n$ is an isomorphism onto and the theorem is established.

5. **Lemmas on homology.** Let $A$ and $B \subset A$ be any arcwise connected spaces. Then we have:

LEMMA 2. *If the injection homomorphism $i_1: \pi_1(B) \to \pi_1(A)$ is an isomorphism (into) then $\pi_2(A, B)$ is Abelian.*

Consider the homotopy sequence

$$\pi_2(A) \xrightarrow{1} \pi_2(A, B) \xrightarrow{2} \pi_1(B) \xrightarrow{3} \pi_1(A).$$

If (3) is an isomorphism, then (2) is into zero and (1) is onto. Therefore the lemma follows from the fact that $\pi_2(A)$ is Abelian.

LEMMA 3. *If $\pi_1(A) = 1$ then the natural homomorphism $\pi_2(A, B) \to H_2(A, B)$ is onto and its kernel is the commutator subgroup of $\pi_2(A, B)$. If also $\pi_1(B) = 1$ and $H_r(A, B) = 0$ for $r = 1, 2, \cdots, n-1$ $(n \geqq 2)$, then the natural homomorphism $\pi_n(A, B) \to H_n(A, B)$ is an isomorphism onto.*

This is an extension of a theorem due to W. Hurewicz, to which it reduces in case $B$ is a single point. S. Eilenberg [**12**, p. 443] has given a proof of Hurewicz's theorem which, with minor modifications, establishes Lemma 3.

On combining Lemmas 2 and 3 we have the following lemma.

[15] These are inner automorphisms if $n = 1$ and are due to the operators in $\pi_1(X)$, $\pi_1(Y)$ if $n > 1$.

[16] Cf. (7.4) below.

LEMMA 4. *If* $\pi_1(A)=1$, $\pi_1(B)=1$, $H_n(A, B)=0$ *for every value of* $n$, *then* $\pi_n(A, B)=0$ *for each* $n$.

6. **Proof of Theorem 3.** Let $f: X\to Y$ be a map which induces isomorphisms of $\pi_1(X)$ and $H_n(\tilde{X})$ $(n=0, 1, \cdots)$ onto $\pi_1(Y)$ and $H_n(\tilde{Y})$ and let the notation be the same as in §3. Let $\tilde{Z}$ be the universal covering space of $Z$, with $x_0\in X$ as base point. Since $f_1:\pi_1(X)\to\pi_1(Y)$ is an isomorphism onto it follows from an argument used at the beginning of §3 that $i_1:\pi_1(X)\to\pi_1(Z)$ is an isomorphism onto. Therefore we may identify $\tilde{X}$, and similarly $\tilde{Y}$, with the sub-sets of $\tilde{Z}$ which cover $X$ and $Y$ respectively. Then it follows from arguments similar to those at the beginning of §3, including (3.1), with $X$, $Y$, $Z$ replaced by $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and homotopy groups replaced by homology groups, that all the relative homology groups $H_n(\tilde{Z}, \tilde{X})$ $(n=1, 2, \cdots)$ are zero. Since $\pi_1(\tilde{X})=1$, $\pi_1(\tilde{Z})=1$ it follows from Lemma 4 that $\pi_n(\tilde{Z}, \tilde{X})=0$ for each $n\geqq 1$. Therefore $\pi_n(Z, X)=0$ $(n\geqq 2)$ and $i_1:\pi_1(X)\to\pi_1(Z)$ is onto. Therefore the theorem follows from the proof of the simpler version of Theorem 1, which was mentioned towards the end of §3.

7. **Proof of Theorem 4.** It follows from the conditions (a), (b) of Theorem 4 and from (3.1), with $X$, $Y$, $Z$ replaced by $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and homotopy groups replaced by homology groups, that $H_n(\tilde{Z}, \tilde{X})=0$ for $n=2, \cdots, N-1$. Therefore $\pi_n(\tilde{Z}, \tilde{X})=0$ and hence $\pi_n(Z, X)=0$ for $n=2, \cdots, N-1$. Also $f_1$ is onto, whence $\pi_1(Z, X)=0$, and $f_{N-1}$ is an isomorphism. Therefore the first half of Theorem 4 follows from the proof of Theorem 2.

The second half of Theorem 4 is trivial if $N=2$, since $\tilde{X}$ and $\tilde{Y}$ are simply connected. Therefore we assume that $N\geqq 3$. Using the same notation as before, let $\tilde{P}$ be the universal covering space of $P$, with a base point $p_0\in P^0$. As at the end of §4, let the base points $x_0\in X$, $y_0\in Y$, $q_0\in Q^0$ be such that

$$(7.1)\qquad x_0=\lambda' p_0=gy_0,\qquad y_0=\mu' q_0=fx_0,$$

and let $\tilde{x}_0$, $\tilde{p}_0$, etc., be the base points in $\tilde{X}$, $\tilde{P}$, etc. Let $\tilde{\lambda}':\tilde{P}\to\tilde{X}$ be the map which *covers* $\lambda':P\to X$, meaning that $\tilde{\lambda}'\tilde{p}_0=\tilde{x}_0$ and $\lambda' p=p\tilde{\lambda}'$, where $p$ denotes both projections $p:\tilde{P}\to P$, $p:\tilde{X}\to X$. According to a remark in §1 the map $\tilde{\lambda}'$ has a right homotopy inverse, $\tilde{\lambda}:\tilde{X}\to\tilde{P}$.

Let $h_0, h_1:\tilde{X}\to A$ be maps of $\tilde{X}$ in any space, $A$, and let

$$(7.2)\qquad h_0\tilde{\lambda}'\mid\tilde{P}^{N-1}\simeq h_1\tilde{\lambda}'\mid\tilde{P}^{N-1},$$

where $\tilde{P}^n$ is the $n$-section of $\tilde{P}$. Let $\phi:K\to\tilde{X}$ be a given map, where

$K$ is a polyhedron and $\dim K \leqq N-1$. Then $\tilde{\lambda}\phi: K \to \tilde{P}$ is homotopic to a map, $\phi'$, such that $\phi' K \subset \tilde{P}^{N-1}$. Since $\tilde{\lambda}'\tilde{\lambda} \simeq 1$ we have $h_i\phi \simeq h_i\tilde{\lambda}'\tilde{\lambda}\phi \simeq h_i\tilde{\lambda}'\phi'$ $(i=0, 1)$. Since $\phi' K \subset \tilde{P}^{N-1}$ it follows from (7.2) that $h_0\tilde{\lambda}'\phi' \simeq h_1\tilde{\lambda}'\phi'$ and hence that $h_0\phi \simeq h_1\phi$. Therefore

$$h_0 \simeq_{N-1} h_1, \tag{7.3}$$

in consequence of (7.2).

Let $f: X \to Y$ and $g: Y \to X$ be such that $gf \simeq_{N-1} 1$, $fg \simeq_{N-1} 1$ and let $\tilde{f}: \tilde{X} \to \tilde{Y}$, $\tilde{g}: \tilde{Y} \to \tilde{X}$ be the maps which cover $f$, $g$. Since $gf \simeq_{N-1} 1$ there is a homotopy, $\theta_t: P^{N-1} \to X$, such that $\theta_0 = gf\lambda' \mid P^{N-1}$, $\theta_1 = \lambda' \mid P^{N-1}$. It follows from (7.1) that $\theta_0 p_0 = \theta_1 p_0 = x_0$. Let $\theta_t$ be lifted into the homotopy, $\tilde{\theta}_t: \tilde{P}^{N-1} \to \tilde{X}$, such that $\tilde{\theta}_0\tilde{p}_0 = \tilde{x}_0$, $\theta_t p = p\tilde{\theta}_t$. It follows from (7.1) that $\tilde{\theta}_0 = \tilde{g}\tilde{f}\tilde{\lambda}' \mid \tilde{P}^{N-1}$ and that $\tilde{\theta}_1 = u\tilde{\lambda}' \mid \tilde{P}^{N-1}$, where $u: \tilde{X} \to \tilde{X}$ is a transformation in the covering group. Hence it follows from (7.2) and (7.3), with $A = \tilde{X}$, $h_0 = \tilde{g}\tilde{f}$, $h_1 = u$, that $\tilde{g}\tilde{f} \simeq_{N-1} u$. Similarly $\tilde{f}\tilde{g} \simeq_{N-1} v$, where $v: \tilde{Y} \to \tilde{Y}$ is in the group of covering transformations. Let $\bar{g} = u^{-1}\tilde{g}$. Then $\bar{g}\tilde{f} \simeq_{N-1} 1$ and

$$\begin{aligned} \tilde{f}\bar{g} &= (v^{-1}v)\tilde{f}\bar{g} \simeq_{N-1} (v^{-1}\tilde{f}\tilde{g})\tilde{f}\bar{g} \\ &= v^{-1}\tilde{f}(\tilde{g}\tilde{f})\bar{g} \simeq_{N-1} v^{-1}\tilde{f}u\bar{g} \\ &= v^{-1}\tilde{f}\tilde{g} \simeq_{N-1} 1. \end{aligned} \tag{7.4}$$

Therefore $\tilde{f}$ is an $(N-1)$-homotopy equivalence.

Since $\tilde{\lambda}'\tilde{\lambda} \simeq 1$ it follows that any (singular) cycle in $\tilde{X}$ is homologous to a continuous[17] cycle. Similarly any cycle in $\tilde{Y}$ is homologous to a continuous cycle. Therefore the homomorphism $H_n(\tilde{X}) \to H_n(\tilde{Y})$, which is induced by $\tilde{f}$, is obviously an isomorphism onto if $n \leqq N-1$. This completes the proof.

8. **Two lemmas.** Let $e^n$ $(n \geqq 1)$ be an $n$-cell, which is an open subset of a Hausdorff space, $A$, and let $A_0 = A - e^n$. Let $\bar{e}^n$, the closure of $e^n$, be the image of an $n$-simplex, $\sigma^n$, in a map, $\phi: (\sigma^n, \dot{\sigma}^n) \to (\bar{e}^n, \bar{e}^n \cap A_0)$, such that $\phi \mid (\sigma^n - \dot{\sigma}^n)$ is a homeomorphism onto $e^n$. Let $B$ and $B_0 \subset B$ be arcwise connected spaces such that $\pi_n(B, B_0) = 0$ and let $f_0: (A, A_0) \to (B, B_0)$ be a given map.

LEMMA 5. *There is a homotopy,* $f_t: (A, A_0) \to (B, B_0)$, rel. $A_0$, *such that* $f_1 A \subset A_0$.

Since $\pi_n(B, B_0) = 0$ there is a homotopy, $\psi_t: (\sigma^n, \dot{\sigma}^n) \to (B, B_0)$, rel.

[17] For example, to the image under $\tilde{\lambda}'$ of a continuous cycle in $\tilde{P}$. By a continuous, $n$-dimensional cycle is meant the image in a map, $K^n \to \tilde{X}$, of a cycle carried by an $n$-dimensional complex $K^n$.

$\dot{\sigma}^n$, such that $\psi_0 p = f_0\phi p$ $(p \in \sigma^n)$, $\psi_1\sigma^n \subset B_0$. Let $g_t:\bar{e}^n \to B$ be given by $g_t = \psi_t\phi^{-1}$. Clearly $\phi^{-1}e^n$, and hence $g_t | e^n$, is single-valued. If $a \in \bar{e}^n - e^n = \bar{e}^n \cap A_0$, then $\phi^{-1}a \subset \dot{\sigma}^n$ and $g_t a = \psi_0\phi^{-1}a = f_0\phi\phi^{-1}a = f_0 a$. Therefore $g_t$ is single-valued and hence continuous (see [8, §5]). Moreover $g_t | \bar{e}^n \cap A_0 = f_0 | \bar{e}^n \cap A_0$. Therefore the requirements of the lemma are satisfied by $f_t$, which is given by $f_t | A_0 = f_0$, $f_t | \bar{e}^n = g_t$.

Let $A$ be a closed subset of a separable metric space, $A'$, let $B$ be a separable metric ANR and let $f_0: A \to B$ have an extension $f_0': A' \to B$. Then the homotopy, $f_t$, of Lemma 5 can be extended to a homotopy $f_t': A' \to B$.

Now let $B$ be a separable, metric ANR and let the homomorphism, $i_n: \pi_n(B_0) \to \pi_n(B)$, which is induced by the identical map $i: B_0 \to B$, be an isomorphism onto for $n = 1, \cdots, m$. Then it follows from (3.1), with $X$, $Z$ replaced by $B_0$, $B$, that $\pi_n(Z, X) = 0$ for $n = 1, \cdots, m$. Let $f_0, f_1: K^m \to B_0$ be maps of an $m$-dimensional, simplicial complex, $K^m$, into $B_0$. Then we have:

LEMMA 6. *If* $if_0 \simeq if_1$ *(in* $B$*) then* $f_0 \simeq f_1$ *(in* $B_0$*).*

Let $g_t: K^m \to B$ be a deformation of $g_0 = if_0$ into $g_1 = if_1$ and let $g: K^m \times I \to B$ be given by $g(x, t) = g_t x$ $(x \in K^m)$. Let $C_n = (K^m \times 0) \cup (K^{n-1} \times I) \cup (K^m \times 1)$. Since $\pi_n(B, B_0) = 0$ $(n = 1, \cdots, m)$ it follows from repeated applications of Lemma 5, with $\bar{e}^n = \sigma_i^{n-1} \times I$, where $\sigma_1^{n-1}, \sigma_2^{n-1}, \cdots$ are the $(n-1)$-simplexes in $K^m$, that $g$ is homotopic, rel. $(K^m \times 0) \cup (K^m \times 1)$, to a map $g': K^m \times I \to B$, such that $g' C_m \subset B_0$. Let $E_i^{m+1} = \sigma_i^m \times I$ $(i = 1, 2, \cdots)$. Then $g' \dot{E}_i^{m+1} \subset B_0$ and $g' | \dot{E}_i^{m+1}$ is contractible in $B$. Since $i_m: \pi_m(B_0) \to \pi_m(B)$ is an isomorphism it follows that $g' | \dot{E}_i^{m+1}$ is contractible in $B_0$. Therefore there is a map $h_i: E^{m+1} \to B_0$ such that $h_i p = g' p$ if $p \in \dot{E}_i^{m+1}$. Let $f: K^m \times I \to B_0$ be given by $fp = g'p$ if $p \in C_m$, $fp = h_i p$ if $p \in E_i^{m+1}$. Then $f_t: K^m \to B_0$, given by $f_t x = f(x, t)$, is a homotopy of $f_0$ into $f_1$, which completes the proof.

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# TOPOLOGICAL CHARACTERIZATION OF FIELDS WITH VALUATIONS

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1. **Introduction.** A topological field is a (commutative) field which is also a topological space satisfying the separation axiom $T_0$, and in which addition, subtraction and multiplication are continuous, two-variable functions. For our purposes it is unnecessary to assume that division is continuous.

If $F$ is any field, topological or not, we define a (nonarchimedean) valuation of $F$ to be a function $v$ on $F$ to a linearly ordered group $\Gamma$ with the symbol 0 adjoined, such that

$$v(xy) = v(x)v(y), \tag{1}$$

$$v(x+y) \leqq \max\,[v(x), v(y)], \tag{2}$$

$$v(x) = 0 \text{ if and only if } x = 0, \tag{3}$$

for all $x$, $y$ of $F$. It is understood that for every $\gamma$ of $\Gamma$, $0<\gamma$ and $0\gamma=\gamma 0=0$. Such a valuation of a field defines a topology, with respect to which $F$ is a topological field, when we specify that the neighborhoods of 0 in $F$ are the sets $U(\gamma)=[x\in F \mid v(x)<\gamma]$, one for each $\gamma$ in $\Gamma$. If $F$ was a topological field to begin with and the topology defined by the valuation is the same as the original topology of $F$, we say that the valuation preserves the topology of $F$.

The question we intend to answer is, "Which topological fields have valuations preserving their topologies?"

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