NOTE ON A THEOREM DUE TO BORSUK

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1. Introduction. Let A, $B\subset A$ and B' be compacta, which are[1] ANR's (absolute neighbourhood retracts). Let $B'\subset A'$ where A' is a compactum, and let $f:(A, B)\to(A', B')$ be a map such that $f|(A-B)$ is a homeomorphism onto $A'-B'$. Thus A' is homeomorphic to the space defined in terms of A, B, B' and the map $g=f|B$ by identifying each point $b\in B$ with $gb\in B'$. K. Borsuk [3] has shown that A' is locally contractible. It is therefore an ANR if dim $A'<\infty$. The main purpose of this note is to prove, without this restriction on dim A':

THEOREM 1. *A' is an ANR.*

We also derive some simple consequences of this theorem. For example, it follows that the homotopy extension theorem, in the form in which the image space is arbitrary, may be extended[2] from maps of polyhedra to maps of compact ANR's, P and $Q\subset P$. That is to say, if $f_0:P\to X$ is a given map, the space X being arbitrary, and if $g_t:Q\to X$ is a deformation of $g_0=f_0|Q$, then there is a homotopy $f_t:P\to X$, such that $f_t|Q=g_t$. For let $R=(P\times 0)\cup(Q\times I)\subset P\times I$ and let $h:R\to X$ be given by $h(p, 0)=f_0p$, $h(q, t)=g_tq$ $(p\in P, q\in Q)$. Since $Q\times I$ is (obviously) a compact ANR it follows from Theorem 1, with $A=Q\times I$, $B=Q\times 0$, $B'=P\times 0$, $A'=R$ that R is an ANR. Therefore R is a retract of some open set $U\subset P\times I$. If $\theta: U\to R$ is a retraction, then $h\theta: U\to X$ is an extension of $h:R\to X$ throughout U. This is all we need for the homotopy extension theorem (see [5, pp. 86, 87]). Thus we have the corollary:

COROLLARY. *A given homotopy, $g_t:Q\to X$, of $g_0=f_0|Q$, can be extended to a homotopy, $f_t:P\to X$, where P and $Q\subset P$ are compact ANR's and $f_0:P\to X$ is a given map of P in an arbitrary space X.*

We also use Theorem 1 to prove another theorem. We shall describe a map $\xi:X\to Y$ as a *homotopy equivalence* if, and only if, there is a map, $\eta: Y\to X$, such that $\eta\xi\simeq 1$, $\xi\eta\simeq 1$, where X and Y are any two spaces. Thus the statement that $\xi:X\to Y$ is a homotopy equivalence implies that X and Y are of the same homotopy type. Let A, B, A', B' and $f:(A, B)\to(A, B')$ be as in Theorem 1 and let $g=f|B$.

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[1] For an account of these spaces, on which this note is based, see [2]. Numbers in brackets refer to the references cited at the end of the paper.

[2] Cf. [4].

Then we shall prove:

THEOREM 2. *If $g:B\to B'$ is a homotopy equivalence so is $f:A\to A'$.*

For example B' may consist of a single point, in which case we describe the identification of B with B' as the operation of *shrinking B into a point.* Then it follows from Theorem 2 that any (compact) absolute retract, $B\subset A$, may be shrunk into a point, without altering the homotopy type of A. As another example let A and B' be cell complexes[3] and B a sub-complex of A. Then A' is also a cell complex, subject to suitable conditions on the map[4] $g=f|B$, and Theorem 2 shows that certain combinatorial operations do not alter the homotopy type of A. For example, if B is the n-section of A and if B' is any complex, of at most n dimensions, which is of the same homotopy type as B, then there is a complex, A', of the same homotopy type as B, whose n-section is B'.

2. **Another theorem.** We prove Theorem 1 by means of another theorem. Let X and $Y\subset X$ be compacta and let Y be an ANR. Given $\rho>0$ let $V_\rho\subset X$ be the subset consisting of points whose distances from Y are less than ρ. We assume that

(a) *given $\epsilon>0$ there is a $\rho(\epsilon)>0$ and an ϵ-homotopy, $\theta_t:X\to X$, such that $\theta_0=1$, $\theta_t|Y=1$, $\theta_1 V_{\rho(\epsilon)}=Y$,*

(b) *given ϵ, $\rho>0$ there is a $u(\epsilon,\rho)>0$ such that any partial realization, $g:L\to X-V_\rho$, whose mesh does not exceed $u(\epsilon,\rho)$, of a finite simplicial complex, K, can be extended to a full realization, $f:K\to X$, whose mesh does not exceed ϵ, $u(\epsilon,\rho)$ being independent of K and L.*

Then we prove:

THEOREM 3. *Subject to these conditions X is an ANR.*

For this we shall need a sharpened form of the homotopy extension theorem. Let P and $Q\subset P$ be compacta and let $f_0:P\to M$ be a given map of P in a metric space M. Let $g_t:Q\to M$ be an ϵ-deformation of $g_0=f_0|Q$. Assume that either

(1) *M is a (separable) ANR or that*

(2) *P is a finite polyhedron and Q a sub-polyhedron.*

Then we have:

LEMMA 1. *Given $\epsilon'>0$ there is an $(\epsilon+\epsilon')$-deformation, $f_t:P\to M$, such that $f_t|Q=g_t$.*

[3] That is, a complex of the sort defined in [6], and in a forthcoming book by S. Eilenberg and N. E. Steenrod.

[4] For example, $gB^n\subset B'^n$ for each $n=0, 1, \cdots$, where K^n denotes the n-section of a complex, K, or $A^n\subset B$, $gB\subset B'^n$ for a particular value of n.

By way of proof it is sufficient to add a few comments to a standard proof of the homotopy extension. (See [5, pp. 86, 87].) Let $R=(P\times 0)\cup(Q\times I)\subset P\times I$ and let $h:R\to M$ be given by $h(p, 0)=f_0p$, $h(q, t)=g_tq$ $(p\in P,\ q\in Q)$. If P is a polyhedron and Q a sub-polyhedron, then R is a polyhedron and hence a neighbourhood retract of $P\times I$ (in fact R is a deformation retract of $P\times I$). Therefore h can be extended throughout some neighbourhood, $U\subset P\times I$, of R, as it can be if M is an ANR and P, Q arbitrary compacta. There is a neighbourhood, $V\subset P$, of Q such that $V\times I\subset U$. Since Q is compact we may take V to be the neighbourhood given by $\delta(p, Q)<\rho$, for some $\rho>0$, where $\delta(p, p')$ is a distance function in P. On following the argument given by Hurewicz and Wallman [5, pp. 86, 87] it is easily seen that the extension $f_t:P\to M$ is an $(\epsilon+\epsilon')$-deformation provided ρ is sufficiently small.

We now proceed to prove Theorem 3 by showing that X is LC^*, as defined by Lefschetz.[5] Given $\epsilon>0$ let $\eta'=\eta(\epsilon/2)/4$, $\rho'=\rho(\eta')/2$, where η is an extension function[5] for Y and $\rho(\eta')$ means the same as in the condition (a). Let

$$\xi_1(\epsilon) = \min\,(2\eta',\ \rho').$$

We shall prove that

$$\xi(\epsilon) = u\{\xi_1(\epsilon),\ \rho'\}$$

is an extension function for X. Let K be a finite simplicial complex and $L\subset K$ a sub-complex, which contains all the vertices of K. Let $g:L\to X$ be a partial relization of K, whose mesh does not exceed $\xi(\epsilon)$. We first assume that $s\subset L$ if $g(s\cap L)\subset X-V_{\rho'}$, where s is the closure of any simplex in K. Let $K_1\subset K$ be the sub-complex consisting of all the (closed) simplexes, $s\in K$, such that $g(s\cap L)$ meets $V_{\rho'}$. Then $K=K_1\cup L$. Let $L_1=K_1\cap L$, $g_1=g|L_1$. Then it is sufficient to prove that g_1 can be extended to a full realization, $f_1:K_1\to X$, whose mesh does not exceed ϵ. For since $K_1\cap L=L_1$, $f_1|L_1=g|L_1$, the desired realization, $f:K\to X$, will be given by $f|L=g$, $f|K_1=f_1$. Clearly $\xi(\epsilon)\leqq\xi_1(\epsilon)$ and we shall prove this special case on the weaker assumption that the mesh of $g:L\to X$ does not exceed $\xi_1(\epsilon)$.

Since $\xi_1(\epsilon)\leqq\rho'$ we have $g_1L_1\subset V_{2\rho'}=V_\rho$ where $\rho=\rho(\eta')$. Let $\theta_t:X\to X$ be the η'-deformation associated with V_ρ as in condition (a). Since $K_1^0\subset K^0\subset L$, $K_1^0\subset K_1$, we have $K_1^0\subset L_1$. Also $\theta_1V_\rho\subset Y$. Therefore $\theta_1g_1:L_1\to Y$ is a partial realization of K_1 in Y, whose mesh does not exceed

[5] [2, pp. 82, 83, 84] (N.B. $K^0\subset L$).

$$\xi_1(\epsilon) + 2\eta' \leqq 2^{-1}\eta(\epsilon/2) + 2^{-1}\eta(\epsilon/2) = \eta(\epsilon/2).$$

Therefore $\theta_1 g_1 : L_1 \to Y$ can be extended to a full realization, $f_0 : K_1 \to Y$, whose mesh does not exceed $\epsilon/2$. By Lemma 1 there is an $(\eta'+\epsilon/8)$-homotopy, $f_t : K_1 \to X$, such that $f_t | L_1 = \theta_{1-t} g_1$. Clearly $\eta(\epsilon/2) \leqq \epsilon/2$, whence $\eta' + \epsilon/8 \leqq \epsilon/8 + \epsilon/8 = \epsilon/4$. Therefore the mesh of $f_1 : K_1 \to X$ does not exceed $\epsilon/2 + 2(\eta' + \epsilon/8) \leqq \epsilon$ and $f_1 | L_1 = \theta_0 g_1 = g_1$. Therefore this special case is established.

In general let $K_0 \subset K$ be the sub-complex consisting of all the closed simplexes, $s \in K$, such that $g(s \cap L) \subset X - V_{\rho'}$. Let $L_0 = K_0 \cap L$. Then $g | L_0$ is a partial realization of K_0, whose mesh does not exceed $\xi(\epsilon) = u\{\xi_1(\epsilon), \rho'\}$. By condition (b) it can be extended to a full realization, $f_0 : K_0 \to X$, of mesh at most $\xi_1(\epsilon)$. Since $K_0 \cap L = L_0$, $f_0 | L_0 = g | L_0$, a map, $g' : K_0 \cup L \to X$, is defined by $g' | K_0 = f_0$, $g' | L = g$ and its mesh does not exceed $\xi_1(\epsilon)$. Therefore L may be replaced by $K_0 \cup L$ and the theorem follows from what we have already proved.

3. **Proof of Theorem** 1. We shall prove Theorem 1 by showing that the conditions (a) and (b) in §2 are satisfied when $X = A'$, $Y = B'$. Let $\delta(a_1, a_2)$ be a distance function in A and let $\epsilon > 0$ be given. Since A is compact there is a $\lambda(\epsilon) > 0$ such that $\delta'(fa_1, fa_2) < \epsilon$ provided $\delta(a_1, a_2) < \lambda(\epsilon)$, where $\delta'(a_1', a_2')$ is a distance function in A'. Let $U_\gamma \subset A$ be the neighbourhood of B which consists of all points, $a \in A$, such that $\delta(a, B) < \gamma$. As shown by Borsuk [3], there is a homotopy, $\phi_t : \overline{U}_\gamma \to A$, such that $\phi_0 = 1$, $\phi_t | B = 1$, $\phi_1 \overline{U}_\gamma = B$ for some $\gamma > 0$. By uniform continuity there is a $\mu > 0$ ($\mu \leqq \gamma$) such that $\delta(\phi_t a, b) = \delta(\phi_t a, \phi_t b) < \lambda(\epsilon)/4$ if $\delta(a, b) \leqq \mu$. Hence $\phi_t | \overline{U}_\mu$ is a $\lambda(\epsilon)/2$-deformation. By Lemma 1, $\phi_t | \overline{U}_\mu$ can be extended to a $\lambda(\epsilon)$-deformation $\psi_t : A \to A$ ($\psi_0 = 1$). Let $\theta_t : A' \to A'$ be given by $\theta_t | B' = 1$, $\theta_t | fA = f\psi_t f^{-1} | fA$. Since $f^{-1} | (A' - B')$ is single-valued and since $f^{-1}B' = B$ and $\psi_t | B = 1$ it follows that θ_t is single-valued. It is therefore continuous.[6] Since $\theta_t | B' = 1$ and the diameter of the trajectory, $\psi_t a$, of any point $a \in A$ is less than $\lambda(\epsilon)$ it follows that θ_t is an ϵ-deformation. Also $\theta_1(fU_\mu) = f\psi_1 U_\mu = fB \subset B'$. Therefore $\theta_1(B' \cup fU_\mu) = B'$. Since $f | (A - B)$ is a homeomorphism onto $A' - B'$ and $fB \subset B'$ it follows that $B' \cup fU_\mu$ is an open subset of A'. For

$$\begin{aligned} f(A - U_\mu) &= f\{(A - B) - (U_\mu - B)\} \\ &= A' - B' - f(U_\mu - B) \\ &= A' - (B' \cup fU_\mu). \end{aligned}$$

But $A - U_\mu$ is compact, whence $f(A - U_\mu)$ is closed and $B' \cup fU_\mu$ open.

[6] See §5 below.

Therefore there is a $\rho(\epsilon)>0$ such that $V_{\rho(\epsilon)}\subset B'\cup fU_{\mu}$, whence $\theta_1 V_{\rho(\epsilon)}=B'$. This establishes (a).

Let $\alpha(\epsilon')$ be an extension function for A. Since $f^{-1}|(A'-B')$ is a homeomorphism and $A'-V_{\rho}$ is a compact subset of $A'-B'$, for a given $\rho>0$, there is a $u(\epsilon,\rho)>0$ such that, if $\delta'(a',a'')<u(\epsilon,\rho)$ $(a',a''\subset A'-V_{\rho})$, then $\delta(f^{-1}a',f^{-1}a'')<\alpha\{\lambda(\epsilon)\}$. If $\psi:L\to A'-V_{\rho}$ is a partial realization, of mesh at most $u(\epsilon,\rho)$, of a complex K, it follows that $f^{-1}\psi:L\to A$ is of mesh at most $\alpha\{\lambda(\epsilon)\}$. The latter can therefore be extended to a full realization, $\phi:K\to A$, of mesh at most $\lambda(\epsilon)$. Then $\phi'=f\phi:K\to A'$ is a realization of K, whose mesh does not exceed ϵ. Moreover $f\phi|L=ff^{-1}\psi=\psi$. Therefore (b) is satisfied and Theorem 1 is established.

4. **Proof of Theorem** 2. We first prove a lemma. Let X, Y be topological spaces[7]: let $X_0\subset X$, $Y_0\subset Y$ be closed subsets and let $\phi:(X,X_0)\to(Y,Y_0)$ be a map such that $\phi|X-X_0$ is a homeomorphism onto $Y-Y_0$. Moreover let the topology of Y be such that a subset $F\subset Y$ is closed if, and only if, $F\cap Y_0$ and $\phi^{-1}F$ are both closed.

LEMMA 2. *If X_0 is a deformation retract*[8] *of X, then Y_0 is a deformation retract of Y.*

After replacing X by a homeomorph, if necessary, we assume that it has no point in common with Y_0 and we unite X, Y_0 in the space, $Q=X\cup Y_0$, of which X and Y_0, each with its own topology, are closed subspaces. Then Y has the identification topology[6] determined by the map $\psi:Q\to Y$, where $\psi|X=\phi$, $\psi|Y_0=1$. Let $\xi_t:X\to X$ be a homotopy such that $\xi_0=1$, $\xi_t|X_0=1$, $\xi_1X=X_0$ and let ξ_t be extended throughout Q by taking $\xi_t|Y_0=1$. Let $\eta_t=\psi\xi_t\psi^{-1}:Y\to Y$. Clearly $\psi^{-1}|Y-Y_0$ is single-valued. Also $\psi^{-1}Y_0=X_0\cup Y_0$. Since $\xi_t|X_0\cup Y_0=1$ it follows that η_t is single-valued and hence continuous.[5] Obviously $\eta_0=1$, $\eta_t|Y_0=1$, $\eta_1Y=Y_0$, which establishes the lemma.

Notice that the topology of Y certainly satisfies the above condition if X is compact (that is, bi-compact) and if Y is a Hausdorff space. For let this be so and let $F\subset Y$ be such that $\phi^{-1}F$ and $F\cap Y_0$ are both closed. Then $\phi^{-1}F$ is compact, whence $\phi\phi^{-1}F$ is also compact and hence closed. But $\phi\phi^{-1}F=F\cap\phi X$ and

$$F=(F\cap\phi X)\cup(F\cap Y_0),$$

[7] We do not need to assume that X and Y satisfy any separation axioms.

[8] Following Lefschetz [1, p. 40] we do not admit that X_0 is a deformation retract of X unless there is a retracting deformation throughout which each point of X_0 is held fixed (see [7]).

whence F is closed. The converse follows from the continuity of ϕ and the fact that Y_0 is closed.

We now turn to Theorem 2. We recall that $f:(A, B)\to(A', B')$ is such that $f|(A-B)$ is a homeomorphism onto $A'-B'$ and $g=f|B$ is a homotopy equivalence. Replacing A, A' by homeomorphs, if necessary, we assume that no two of the spaces A, A', $A\times I$ have a point in common. We form the mapping cylinder, Γ, of the map f by identifying $(a, 0)\in A\times I$ with a and $(a, 1)$ with $fa\in A'$ for each[9] $a\in A$. The theorem will follow when we have proved that A is a deformation retract[10] of Γ.

Let $C=(A\times 0)\cup(B\times I)$. Then C is an ANR, as shown in §1. Let $\delta_s:A\times I\to A\times I$ be the retracting deformation of $A\times I$ onto $A\times 0$, which is given by $\delta_s(a, t)=(a, t-st)$ $(0\leqq s\leqq 1)$. Then $\delta_s C\subset C$ and it follows that C is a deformation retract[10] of $A\times I$. Let $\phi:A\times I\to\Gamma$ be the map which is given by $\phi(a, 0)=a$, $\phi(a, 1)=fa$, $\phi(a, t)=(a, t)$ if $0<t<1$. Since $fB\subset B'$ and $f|(A-B)$ is a homeomorphism onto $A'-B'$ it follows that $\phi|(A\times I)-(B\times 1)$ is a homeomorphism onto $\Gamma-B'$. Therefore $\phi|(A\times I)-C$ is a homeomorphism onto $\Gamma-(B'\cup\phi C)$. It follows from Lemma 2 that $B'\cup\phi C$ is a deformation retract of Γ. Since $g=f|B$ is a homotopy equivalence, B is a deformation retract[10] of $\Gamma_g=B'\cup\phi(B\times I)$, which is the mapping cylinder of $g:B\to B'$. A homotopy, $\eta_s:\Gamma_g\to\Gamma_g$, such that $\eta_0=1$, $\eta_s|B=1$, $\eta_1\Gamma_g=B$, can be extended throughout $B'\cup\phi C=B'\cup\phi(B\times I)\cup A$ by writing $\eta_s|A=1$. The result is a retracting deformation of $B'\cup\phi C$ onto A. Therefore A is a deformation retract of $B'\cup\phi C$ and hence of Γ, which proves the theorem.

5. **Note on identification spaces.**[11] Let $\phi:P\to X$ be a map of a space P onto a space X, which has the *identification topology* determined by ϕ. That is to say a subset $X_0\subset X$ is closed (open) if, and only if, $\phi^{-1}X_0$ is closed (open). A subset $P_0\subset P$ is said to be *saturated* (with respect to ϕ) if, and only if, $P_0=\phi^{-1}\phi P_0$. Therefore $X_0\subset X$ is closed if, and only if, it is the image under ϕ of a saturated closed set $P_0=\phi^{-1}X_0$. If P is compact and if X is a Hausdorff space then X certainly has the identification topology determined by ϕ. For in this case, if $P_0\subset P$ is closed, and hence compact, ϕP_0 is compact, and hence closed, whether P_0 is saturated or not.

Let $f:P\to Z$ be a map of P in any space Z.

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[9] The points in A and A' shall retain their individualites in Γ, so that $A, A'\subset\Gamma$.

[10] See [7, Theorems 1.4 and 3.7] and [8].

[11] Cf. [9, pp. 61 et seq.] and [10, pp. 52 et seq.]. Concerning the theorem on p. 56 of [10] and Lemma 4 below see the correction at the beginning of [11].

LEMMA 3. *If X has the identification topology determined by ϕ, then the transformation $f\phi^{-1}:X\to Z$ is continuous if it is single-valued.*

If $p\in P$, then $p\in\phi^{-1}\phi p$, whence $fp\in f\phi^{-1}\phi p$. If $f\phi^{-1}$ is single-valued it follows that $fp=f\phi^{-1}\phi p$, or that $(f\phi^{-1})\phi=f$. Therefore the lemma follows from Theorem 1 on p. 53 of [**10**].

Let X have the identification topology determined by $\phi:P\to X$ and let $h:P\times I\to X\times I$ be given by $h(p, t)=(\phi p, t)$ $(p\in P,\ 0\leq t\leq 1)$. Then it follows from Lemma 4 below that $X\times I$ has the identification topology determined by h. Therefore we have the following corollary to Lemma 3, with P, X, ϕ and f replaced by $P\times I$, $X\times I$, h and $f:P\times I\to Z$, where $f(p, t)=f_t p$.

COROLLARY. *If $f_t:P\to Z$ is a given homotopy in any space, Z, then $f_t\phi^{-1}:X\to Z$ is continuous if it is single-valued.*

Let $\psi:Q\to Y$ be a map of a space, Q, onto a space, Y, which has the identification topology determined by ψ and which satisfies the following condition. Each point in any saturated open set, $V\subset Q$, is contained in a saturated open set, whose closure is a compact subset of V. This condition is satisfied if, for example, Q and Y are compacta. For in this case, if $q\in V$, there is a neighbourhood, $W\subset Y$, of ψq, such that $\overline{W}\subset\psi V$. Then $\psi^{-1}W$ is a saturated open set, whose (compact) closure is contained in V. In particular the condition is satisfied if $Q=Y=I$ and $\psi=1$.

Let X, Y have the identification topologies determined by maps $\phi:P\to X$, $\psi:Q\to Y$, which are onto X and Y, and let Y satisfy the above condition. Let $h:P\times Q\to X\times Y$ be given by $h(p, q)=(\phi p, \psi q)$ $(p\in P,\ q\in Q)$. Then we have:

LEMMA 4. *The space $X\times Y$ has the identification topology determined by h.*

Let $W\subset P\times Q$ be an open subset, which is saturated with respect to h, and let (x_0, y_0) be an arbitrary point in hW. Then we have to prove that there are open sets $U\subset P$, $V\subset Q$, which are saturated with respect to ϕ, ψ and are such that

$$(x_0, y_0)\in\phi U\times\psi V\subset hW.$$

Let $p_0\in\phi^{-1}x_0$, $q_0\in\psi^{-1}y_0$ and let

$$(p_0\times Q)\cap W = p_0\times Q_0.$$

Then it is easily verified that Q_0 is an open subset of Q, which is saturated with respect to ψ. Therefore q_0 is contained in a saturated open set, $V \subset Q$, such that $\overline{V}$ is a compact subset of Q_0. Let U be the totality of all points, $p \in P$, such that $p \times \overline{V} \subset W$. Then $p_0 \in U$ and $U \times \overline{V} \subset W$, whence

$$(x_0, y_0) \in \phi U \times \psi V = h(U \times V) \subset hW$$

and the lemma will follow when we have proved that U is a saturated, open subset of P.

If X_0, Y_0 are any subsets of X, Y we have $h^{-1}(X_0 \times Y_0) = \phi^{-1}X_0 \times \psi^{-1} Y_0$, whence

$$\phi^{-1}\phi U \times \overline{V} \subset \phi^{-1}\phi U \times \psi^{-1}\psi \overline{V} = h^{-1}h(U \times \overline{V}) \subset h^{-1}hW = W.$$

Therefore $\phi^{-1}\phi U \subset U$, whence $\phi^{-1}\phi U = U$, that is, U is saturated. Let p be any point in U. Then $p \times \overline{V} \subset W$ and since W is open and $\overline{V}$ is compact it is easily proved that there is an open set, $N \subset P$, such that $p \in N$ and $N \times \overline{V} \subset W$. Therefore $N \subset U$. Therefore U is open and the lemma is established.

References

1. S. Lefschetz, *Algebraic topology*, Princeton, 1942.
2. ———, *Topics in topology*, Annals of Mathematics Studies, Princeton, 1942.
3. K. Borsuk, Fund. Math. vol. 24 (1935) pp. 249–258.
4. Sze-Tsen Hu, C. R. Acad. Sci. URSS. vol. 57 (1947), pp. 231–234.
5. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941.
6. J. H. C. Whitehead, Comment, Math. Helv. vol. 22 (1949).
7. R. H. Fox, Ann. of Math. vol. 44 (1943) pp. 40–50.
8. H. Samelson, Ann. of Math. vol. 45 (1944) pp. 448–449.
9. P. Alexandroff and H. Hopf, *Topologie*, Berlin, 1935, and Ann Arbor, 1945.
10. N. Bourbaki, *Topologie générale*, Chap. I, *Structures topologiques*, Paris, 1940. (Actualités Scientifiques et Industrielles, no. 858.)
11. ———, *Topologie générale*, chap. III, Paris, 1942. (Actualités Scientifiques et Industrielles, No. 916.)

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